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A THREE-DIMENSIONAL TREATMENT OF GROUPS OF LINEAR TRANSFORMATIONS.

By Deborah May Hickey.

Introduction. In his treatment of the theory of linear transformations Professor L. R. Ford * has introduced the concept of the isometric circle. Used in the theory of discontinuous groups it leads easily to the construction of a fundamental region. The resulting simplification in the treatment of the linear transformation suggested the use of the same notion in connection with space transformations.

The kind of space transformations to be studied is defined as follows. Let Σ_0 be the unit sphere with center at the origin; and let $\sigma_1, \sigma_2, \cdots, \sigma_{2p}$ be an even number of spheres orthogonal to Σ_0. Let inversions be made successively in these spheres. Any point P is carried into a point P_1 by the inversion in σ_1; P_1 into P_2 by the inversion in σ_2; and so on, finally P_{2p-1} into P' by the inversion in σ_{2p}. To each point P there corresponds one and only one point P'. Two such sequences of inversions will be considered to define the same transformation if they result in the same P' for each point P of space.

The effect of the preceding transformation on the surface of Σ_0 is to carry it into itself in a one-to-one and directly conformal manner. Conversely, it will appear that each such transformation of the surface of Σ_0 can be achieved by a unique space transformation of the type defined.

By a stereographic projection of Σ_0 on the complex plane there is set up a one-to-one correspondence between the set of space transformations and all linear transformations.

A study is made of groups of these space transformations. It is found that all discontinuous groups of linear transformations correspond to *properly* discontinuous groups in space. The introduction of the isometric sphere leads to the construction of a fundamental region in a very simple manner. The faces, edges, and vertices of this region have striking properties.

The continuous and directly conformal transformation of the surface of a sphere into itself. If the surface of a sphere be mapped in a one-to-one and continuous manner on itself, there results, in general, a magnification of

* *Automorphic Functions*, McGraw-Hill Book Co. (1929). H. Poincaré was the first to introduce a space transformation corresponding to the linear transformation, *Acta Mathematica*, Vol. 3 (1883-1884), pp. 49-92.

some parts and a diminution of other parts. If the mapping is conformal, we have the following

THEOREM 1. *Let the surface of a sphere be transformed in a one-to-one and directly conformal manner on itself. Then, provided the transformation is not a rotation of the sphere about an axis, the locus of points in the neighborhood of which lengths and areas are unaltered is a small circle.*

By a suitable choice of coördinates we can make Σ_0 the sphere in question. We designate the transformation by S. Let Σ_0 be projected stereographically on a z-plane through its center.* Then, as is well known, the plane undergoes a linear transformation when the sphere is transformed by S. Denote by ds the length of an element of arc on the surface of Σ_0 and by ds' the length of the transformed element. We have

$$ds^2 = d\xi^2 + d\eta^2 + d\zeta^2,$$

where $P(\xi, \eta, \zeta)$ is the initial point of the element.

The point P on Σ_0 and its stereographic projection $z = x + iy$ in the plane are connected by the relations

$$\xi = \frac{2x}{z\bar{z} + 1}, \quad \eta = \frac{2y}{z\bar{z} + 1}, \quad \zeta = \frac{z\bar{z} - 1}{z\bar{z} + 1},$$

where $\bar{z}$ is the conjugate imaginary of z; whence

$$ds^2 = \frac{8(dx^2 + dy^2)}{(z\bar{z} + 1)^2}.$$

Similarly

$$ds'^2 = \frac{8(dx'^2 + dy'^2)}{(z'\bar{z}' + 1)^2},$$

where $z' = x' + iy'$ is the transform of z by a linear transformation

$$z' = \frac{az + b}{cz + d}, \quad \text{with} \quad ad - bc = 1.$$

Then since $dx'^2 + dy'^2 = dz'd\bar{z}'$ and $dz'/dz = 1/(cz + d)^2$, the condition that $ds' = ds$ is that

$$(z'\bar{z}' + 1)^2(cz + d)^2(\bar{c}\bar{z} + \bar{d})^2 = (z\bar{z} + 1)^2,$$

which reduces to

$$(1) \qquad (1 - a\bar{a} - c\bar{c})z\bar{z} - (a\bar{b} + c\bar{d})z - (\bar{a}b + \bar{c}d)\bar{z} + (1 - b\bar{b} - d\bar{d}) = 0.$$

* The stereographic projection of the sphere is made by projecting from the point $(0, 0, 1)$ upon the z-plane. The equator of Σ_0 in the z-plane remains fixed.

If $1 - a\bar{a} - c\bar{c} \neq 0$, equation (1) represents a circle C with center and radius respectively

$$p = \frac{\bar{a}b + \bar{c}d}{1 - a\bar{a} - c\bar{c}}, \quad r = (a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} - 2)^{1/2}/(1 - a\bar{a} - c\bar{c}).$$

The expression under the radical sign is non-negative. Moreover, except for a rotation of Σ_0 into itself, it is positive. In fact,

$$a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} - 2 = (a - \bar{d})(\bar{a} - d) + (b + \bar{c})(\bar{b} + c) \geqq 0,$$

the equality sign holding if and only if $a = \bar{d}$, $b = -\bar{c}$. For these relations (1) is identically satisfied. They are the conditions for a rotation of Σ_0 about an axis.

If $1 - a\bar{a} - c\bar{c} = 0$ and $1 - b\bar{b} - d\bar{d} \neq 0$, equation (1) reduces to the equation of a straight line, not passing through the origin.

The projection of C back on Σ_0 is a circle I_S. If, in particular, $1 - a\bar{a} - c\bar{c} = 0$ and $1 - b\bar{b} - d\bar{d} \neq 0$, the circle passes through the north pole. Under the transformation S infinitesimal lengths in the neighborhood of I_S are unaltered; hence infinitesimal areas on the surface are unaltered also.

We call I_S the *isometric circle* of the transformation S.

The circle I_S is the complete locus of points in the neighborhood of which lengths on Σ_0 are unaltered by S. For, any point P on Σ_0 not on I_S has as a projection in the plane a point $Q(z_0)$ which is not on C. For z_0 inside C, the left member of (1) is negative, and $ds'^2 > ds^2$; for z_0 outside C, $ds'^2 < ds^2$.

The isometric circle I_S cannot be a great circle of Σ_0. In fact, the distance Δ of the plane of I_S,

$$\begin{aligned}\xi(a\bar{b} + c\bar{d} + \bar{a}b + \bar{c}d) + i\eta(a\bar{b} + c\bar{d} - \bar{a}b - \bar{c}d)& \\ + \zeta(a\bar{a} - b\bar{b} + c\bar{c} - d\bar{d}) + (a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} - 2) &= 0,\end{aligned}$$

from the center of Σ_0 is given by

$$\Delta = [(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} - 2)/(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} + 2)]^{1/2},$$

which is essentially positive, since the numerator vanishes when and only when $a = d$, $b = -\bar{c}$, that is, for a rotation of Σ_0 about an axis.

The transformation S carries I_S into a circle I'_S of the same size. I'_S is obviously the isometric circle of the inverse transformation S^{-1}.

The surface of the smaller polar cap of Σ_0 cut off by I_S we call the *interior of* I_S; that of the larger polar cap we call the *exterior of* I_S.

THEOREM 2. *Lengths and areas in the interior of* I_S *are increased in magnitude by the transformation* S*; those on the exterior of* I_S *are decreased.*

The caps cut off by I_S are carried into the equal caps cut off by I'_S. Since one cap is magnified and the other diminished, the smaller cap must undergo magnification and the larger diminution, whence the theorem. We have also

THEOREM 3. *The transformation S carries the interior and the exterior of I_S into the exterior and interior respectively of I'_S.*

Relative position of I_S and I'_S. If the linear transformation T corresponding to S is non-loxodromic, we can state the following facts about the position of I_S with respect to I'_S:

THEOREM 4. *If T is elliptic, I_S and I'_S intersect in the fixed points of S; if T is hyperbolic, I_S and I'_S are external; if T is parabolic, I_S and I'_S are tangent at the fixed point of S.*

This arrangement of the isometric circles on Σ_0 can be shown to exist by considering the relation between the fixed points and the fixed circles of the respective types of linear transformations and their projections on Σ_0.

Space transformations. We turn now to the space transformations resulting from successive inversions in an even number of spheres (or planes) orthogonal to Σ_0.

If two such transformations transform three points on Σ_0 in the same way, they transform all points of space in the same way.

If two sequences S and S' transform three points of Σ_0 alike, they transform all points of Σ_0 in the same way. For, when Σ_0 is projected stereographically on the plane there correspond to S and S' two linear transformations which transform three points alike; the two linear transformations are thus identical.

A point P of space, outside Σ_0 say, may be determined by three non-coaxal spheres orthogonal to Σ_0 through P. These intersect Σ_0 in three circles which are carried into the same three circles by S and S'. P is carried by both S and S' into the unique point P' outside Σ_0 lying on the three spheres orthogonal to Σ_0 through the latter three circles.*

Furthermore, there is a space transformation which produces any prescribed one-to-one and directly conformal transformation of Σ_0 on itself.

Consider S and its inverse S^{-1}. If Σ_0 is not rotated, S carries I_S into I'_S. Let Σ_S and Σ'_S be the spheres through these circles respectively, orthogonal to Σ_0. S must also transform Σ_S into Σ'_S of the *same size.*

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* Poincaré, *loc. cit.* 4.

The sphere Σ_S is called the *isometric sphere* of the transformation S. Σ'_S is obviously the isometric sphere of S^{-1}.

Sequences Equivalent to S. We propose next to replace the sequence of inversions defining S by an equivalent system consisting, in general, of an inversion and either one or three reflections.

If S is a rotation it is equivalent to successive reflections in two diametral planes of Σ_0. This case we shall not consider.

Let P be a point on I_S and let P' be the point on I'_S into which S carries P. Since lengths on I_S are unaltered, if I_S be placed on I'_S so that P falls on P' and the orientation is correct, each point on I_S will fall on its corresponding point on I'_S.

As a point P moves counter-clockwise around I_S the corresponding point moves clockwise around I'_S. For, the interior of I_S goes into the exterior of I'_S, and the corresponding point must trace the boundary of the latter region in the same sense as the former is traced. Hence I_S must be turned over before being applied to I'_S.

Any sequence of an even number of inversions in spheres orthogonal to Σ_0, or reflections in diametral planes, is equivalent to S if it carries I_S into I'_S without altering lengths so that P falls on P' and so that the order of points about I'_S is opposite to that of the original points about I_S.

Let Π_S be the plane perpendicular bisector of the line of centers of Σ_S and Σ'_S, if these spheres are not coincident; if they coincide, let Π_S be the plane perpendicular bisector of the line joining any point P on I_S with its corresponding point P' on I'_S.

First, make an inversion in Σ_S. This leaves points on I_S unchanged. Second, make a reflection in Π_S, which reverses the direction of arcs on I_S. Finally, make a rotation about the line of centers of Σ'_S and Σ_0, so that the transform of P on I_S by the reflection in Π_S is brought into coincidence with P'. Then all the points of I_S are carried into their proper positions on I'_S. Since the rotation is equivalent to two reflections in planes, the result of the three operations is to leave angles on Σ_0 unchanged and to transform points on I_S into points on I'_S exactly as S transforms them.

These results may be stated as

THEOREM 5. *The space transformation S, if not a rotation, is equivalent to*

(1) *an inversion in Σ_S, followed by*

(2) *a reflection in Π_S, followed by*

(3) *a rotation through a suitable angle Θ about the fixed axis $0P'_S$ through the centers of Σ'_S and Σ_0.*

Let U_{Π} denote the reflection in Π_S; U_{Σ}, $U_{\Sigma'}$ the inversions in Σ_S, Σ'_S; and U_R, $U_{R'}$ the rotations of Σ_S, Σ'_S respectively, the directions of rotation being opposite. Then the transformations $U_{R'}U_{\Pi}U_{\Sigma'}$, $U_{\Pi}U_RU_{\Sigma'}$, $U_{\Sigma'}U_{\Pi}U_{R'}$, $U_{\Pi}U_{\Sigma}U_{R'}$, $U_{R'}U_{\Sigma'}U_{\Pi'}$, and $U_{\Sigma'}U_{R'}U_{\Pi}$ are all equivalent since they transform three points on Σ_0 in the same way. Hence the sequence of the above theorem may be varied for a given transformation.

Remark. The center of an isometric sphere Σ_S is the transform of ∞ by S^{-1}, the inverse of S. For by the operations of Theorem 5 in the reverse order, ∞ remains fixed until the inversion (1) is performed, when it is carried to the center of Σ_S.

THEOREM 6. *Lengths inside Σ_S are increased in magnitude by the transformation S, those outside Σ_S are decreased.*

Of the geometric operations of Theorem 5 that accomplish S only the inversion in Σ_S changes lengths. Lengths within the sphere of inversion are increased, those without are decreased.

Remark. Any given sphere orthogonal to Σ_0 is the isometric sphere of an infinite number of possible space transformations; for example, the transformation resulting from an inversion in the given sphere, followed by a reflection in any diametral plane of Σ_0.

THEOREM 7. *The distance of a point P from the center of a sphere Σ_1 is unaltered by inversion in a sphere Σ if P lies on the surface of Σ_1 or on that of Σ; it is decreased if P lies outside both Σ_1 and Σ or inside both; it is increased if P lies outside one and inside the other.*

Without loss of generality we can take Σ as the unit sphere with center at the origin and Σ_1 with center $Q(q, 0, 0)$ on the ξ-axis. The spheres are given by

$$\Sigma: \ \xi^2 + \eta^2 + \zeta^2 = 1; \qquad \Sigma': \ (\xi - q)^2 + \eta^2 + \zeta^2 = q^2 - 1.$$

Let $P(\xi, \eta, \zeta)$ be a point in space and $P'(\xi', \eta', \zeta')$ its inverse with respect to Σ. If h, h' denote the distances PQ, $P'Q$ respectively, then

$$h^2 = (\xi - q)^2 + \eta^2 + \zeta^2, \qquad h'^2 = (\xi' - q)^2 + \eta'^2 + \zeta'^2.$$

With the relation $\xi' = \xi/(\xi^2 + \eta^2 + \zeta^2)$ and similar ones for η', ζ', the difference $h^2 - h'^2$ can easily be put into the form

$$h^2 - h'^2 = \frac{(\xi^2 + \eta^2 + \zeta^2 - 1)[(\xi - q)^2 + \eta^2 + \zeta^2 - q^2 + 1]}{(\xi^2 + \eta^2 + \zeta^2)}.$$

The first factor in the numerator is positive for points outside Σ, negative for points inside, and zero for points on Σ. The same is true of the second factor for points with respect to Σ_1. The conclusions of the theorem are then evident.

Now consider the effect of a transformation S on the distance of a point P of space from the center of Σ_0. Of the three geometric operations of S only the inversion in Σ_S can affect this distance. Theorem 7 applied to the orthogonal spheres Σ_S and Σ_0 gives the following

THEOREM 8. *The distance of a point P from the center of Σ_0 is unaltered by S if P lies on either Σ_S or Σ_0; it is decreased if P lies outside or inside both; it is increased if P lies outside one and inside the other.*

Groups of space transformations. A group that contains infinitesimal transformations on Σ_0 is said to be *continuous on* Σ_0; that is, if given $\epsilon > 0$, there exists a transformation S of the group such that the distance between any point P on Σ_0 and its transform P' by S is less than ϵ. Otherwise the group is said to be *discontinuous on* Σ_0.

If for a discontinuous group there exists some point P on Σ_0 in whose neighborhood there is none of its transforms by the group, the group is *properly discontinuous on* Σ_0; if no such point exists, it is *improperly discontinuous on* Σ_0.

A study of groups of space transformations yields first two fundamental properties that characterize the continuous and the discontinuous groups on Σ_0.

THEOREM 9. *If a group contains an infinite number of rotations of Σ_0 into itself, it is continuous on Σ_0.*

Briefly, we can select a sequence of rotations $T_1, T_2, \cdots$, such that the axis of T_n approaches a limiting position and the angle of rotation θ_n approaches a limit. Then the rotation $T^{-1}_{n+1}T_n$, for n sufficiently large, alters the position of every point on Σ_0 by less than a preassigned small amount.

COROLLARY. *A group discontinuous on Σ_0 contains at most a finite number of rotations.*

THEOREM 10. *If for a group the number of isometric spheres of radii exceeding any positive number is infinite, the group is continuous on Σ_0.*

If the radii are bounded, we may select a sequence of transformations $T_1, T_2, \cdots$, where Σ_n approaches a limiting sphere, the sequence Σ'_n of the inverse transformations approaches a limiting sphere, and the angle of rotation of Theorem 5 approaches a limit. Then $T^{-1}_{n+1}T_n$, for n sufficiently large,

changes the position of every point on Σ_0 by less than a preassigned small amount.

If the radii are unbounded, the proof is similar, the limiting positions of Σ_n and Σ'_n being planes.

THEOREM 11. *For a group discontinuous on Σ_0 the number of isometric spheres of radii exceeding any positive number is finite.*

Transformations of a group. It can be shown by a proper choice of a transformation that a group may be transformed so that the resulting group contains no rotations of Σ_0 into itself.

In the following work it is assumed that such a transformation has been made on the group considered.

Limit points of a discontinuous group. Let S_k be the members of the group G discontinuous on Σ_0. Let r_k be the radius of Σ_{S_k}. Then by Theorem 11 any infinite sequence $r_1, r_2, \cdots$ of radii of distinct spheres has $\lim\limits_{i=\infty} r_i = 0$.

A *limit point* of a group is defined to be a cluster point of centers of isometric spheres. Any point not a limit point of the group is called an *ordinary point.*

THEOREM 12. *The limit points of a discontinuous group lie on Σ_0.*

This is immediate, for the center of an isometric sphere whose radius approaches zero must approach the surface of Σ_0 since the spheres are orthogonal to Σ_0.

We say two figures F_1 and F_2 are *congruent with respect to a group* if $T(F_1) = F_2$, where $T \neq 1$ is a transformation of the group.

The region R of the group. A region R_0 is said to be a *fundamental region* of a discontinuous group if

(1) No two points of the region are congruent.

(2) In the neighborhood of any point on the boundary there are points congruent to points of the region.

The *region R* of the group is the space exterior to all isometric spheres of the group. A point P belongs to R if a region about P exists no point of which is interior to an isometric sphere of the group.

THEOREM 13. *No two points of R are congruent by any transformation of the group.*

For any point of R is carried by any member of the group into a point interior to the isometric sphere of the inverse transformation and so outside R.

THEOREM 14. *In the neighborhood of any limit point P of a discontinuous group lie an infinite number of distinct points congruent to any point of space with the possible exception of P and one other limit point.*

The proof of this theorem we omit.

THEOREM 15. *R and the regions congruent to R form a set of regions which extend into the neighborhood of every point of space.*

If this theorem is not true, there exists a closed sphere σ of ordinary points in which is no point of R or point congruent to R.* The points of σ are contained in a finite number of isometric spheres. Let σ' be a subsphere of σ wholly within the isometric spheres of $T_1, \cdots, T_n$ and no others. Consider $\sigma'_1 = T_1(\sigma)$. If a point of σ'_1 is within the isometric sphere of S then ST_1 increases lengths at a point of σ, so $ST_1 = T_k$. It follows that the points of σ' are contained in $n - 1$ isometric spheres at most; namely, those of the transformations $T_2T_1^{-1}, \cdots, T_nT_1^{-1}$. Making S we find a sphere congruent to a subsphere of σ', and hence congruent to a subsphere of σ, lying in $n - 2$ isometric spheres at most. Continuing in this way we arrive at a sphere congruent to a subsphere of σ which lies in no isometric spheres and hence is in R. This contradiction proves the theorem.

THEOREM 16. *R constitutes a fundamental region of the group.*

By Theorem 13 R satisfies the first condition for a fundamental region. In the neighborhood of any point P on the boundary of R there is a point Q within an isometric sphere in whose neighborhood by Theorem 15 are transforms of R. Thus R satisfies the second condition also.

THEOREM 17. *Any closed region not containing limit points of the group can be filled by a finite number of transforms of R, including possibly R itself. These regions fit together without lacunary spaces.*

Let B be the closed region. Since B contains no limit points, the number of isometric spheres containing points of B is finite, for an infinite sequence of spheres would have a cluster point of centers which would necessarily lie in B.

Each point of R is carried by a transformation T of the group into a point of R_T in the interior of Σ'_T. If Σ'_T contains no point of B, then R_T contains no point of B. Thus the number of regions R_T containing points of B is finite.

By Theorem 15 there can exist no lacunary spaces not filled by transforms of R.

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* This method of proof is due to Professor L. R. Ford.

THEOREM 18. *Within any region enclosing a limit point there lie an infinite number of distinct transforms of the entire region R.*

Let P be the limit point and σ be a small sphere with P as center. Let Σ'_T be an isometric sphere lying entirely inside σ. Then $R_T = T(R)$ lies entirely within Σ'_T. Since σ contains an infinite number of isometric spheres, it contains an infinite number of transforms of R.

Groups with more than two limit points. If a group contains more than two limit points, the set of limit points is perfect. Let G be a group discontinuous on Σ_0 with more than two limit points. These points lie on Σ_0. Let P be a point on Σ_0 and U_P a spherical cap enclosing P. If every U_P contains limit points of G, for each P of Σ_0, the set of limit points is everywhere dense on Σ_0.

By Theorem 14 in the neighborhood of P lie an infinite number of transforms of P with two possible exceptions, P and one other limit point. P is exceptional only in that its transforms are not distinct. Thus in U_P there are transforms of P. If the set of limit points is everywhere dense on Σ_0, each point of Σ_0 has in its neighborhood transforms of itself. The group is therefore *improperly discontinuous on* Σ_0.

If the set of limit points is not everywhere dense on Σ_0, there is a closed cap U on Σ_0 consisting of ordinary points. Any point P of U which is not one of the finite number of possible fixed points has no transforms within a suitably small neighborhood of P. Hence a discontinuous group whose set of limit points is not everywhere dense on Σ_0 is *properly discontinuous on* Σ_0.

THEOREM 19. *A necessary and sufficient condition that a group be properly discontinuous on Σ_0 is that the region R contain in its interior a point of Σ_0.*

A point P of Σ_0 within R has no transforms within some cap U_P about P. Conversely, if the group is properly discontinuous on Σ_0, there exists at least one point P on Σ_0 and a sphere σ about P which contains only ordinary points. The sphere σ can be filled by a finite number of transforms of R. Let R_m be one of them. R_m has a finite number of bounding spherical surfaces in σ, each orthogonal to Σ_0. At least one point P' on Σ_0 exists interior to R_m. Hence $T_m^{-1}(P') = P_1$, where P_1 is interior to R and lies on Σ_0.

Thus for a group improperly discontinuous on Σ_0 no point of the surface of Σ_0 belongs to the interior of R. Hence R is divided into two regions, one inside, the other outside Σ_0, having at most common boundary points on Σ_0.

The boundary of R. A point P of the boundary of R is either a limit point or an ordinary point. An ordinary boundary point lies on a finite number of isometric spheres and is inside none.

We consider below the properties of ordinary boundary points of R; those of the limit points have been discussed above.

An ordinary boundary point P belongs to one of the following classes:

(α) P on just one isometric sphere.

(β) P on two or more isometric spheres having a circle in common.

(γ) P on a finite number of isometric spheres not having a circle in common.

(δ) P a point of tangency of two or more isometric spheres and on no isometric spheres not tangent at P.

If P is a fixed point of $S = S^{-1}$ and on no other isometric sphere than Σ_S, it shall be regarded as belonging to class (β).

Points of Class (α). Let P lie on the isometric sphere Σ_S of S. Then $P' = S(P)$ lies on Σ'_S and is distinct from P.

If P' is inside an isometric sphere Σ_U of U, then US increases lengths near P', for S does not alter lengths near P and U increases lengths near P'. Hence P lies inside Σ_{US}, which is contrary to hypothesis. P' is therefore a boundary point of R.

P' belongs to class (α) also. For, if P' were on Σ_U, $U \neq S^{-1}$, then US would leave lengths near P unchanged since S does not alter lengths near P, nor U those near P'. Thus P would be on Σ_{US} also, which is contrary to the fact.

Points on Σ_S in the neighborhood of P are ordinary boundary points. Part of the boundary of R is therefore a portion of the surface of Σ_S and a congruent portion of Σ'_S. The two congruent parts are equal in area since lengths, and consequently areas, on Σ_S are unchanged by S. This part of Σ_S limited by boundary points of other classes is called a *face*.

These results with Theorem 8 give

THEOREM 20. *The boundary points of R of class (α) form sets of spherical faces which are congruent in pairs. The congruent faces are equal in area and congruent points on the faces are equidistant from the center of Σ_0.*

Points of Class (β). Let P lie on C_1, the circle common to the two or more isometric spheres. Let E_1 be the arc (or arcs) of C_1 which consists entirely of ordinary boundary points of R. We call E_1 an *edge* of R.

Let $T_1, T_2, \cdots, T_m$ be the transformation whose isometric spheres pass through E_1. Then $E_k = T_k(E_1)$, $k = 2, \cdots, m$, is an edge congruent to E_1 on the boundary of R, and there are no other congruent edges. That E_k is on the boundary of R follows as in class (α). From the way lengths on E_k are affected we see that $T_k^{-1}, T_1T_k^{-1}, \cdots, T_{k-1}T_k^{-1}, T_{k+1}T_k^{-1}, \cdots, T_mT_k^{-1}$ have

isometric spheres passing through E_k, and that E_k is external to all other isometric spheres.

Since T_k carries E_1 into E_k with no change in length we have

THEOREM 21. *The boundary points of class* (β) *form sets of congruent circular arcs, or edges. All congruent edges are equal in length.*

Points of Class (γ). P of class (γ) is called a *vertex.* Let P_1 be on the isometric sphere Σ_{T_1} which forms a face F_1 of R. Let $T_1(P_1) = P_2$. Since $T_1(F_1) = F'_1$ of the same area and shape, that is, a spherical polygon on Σ'_{T_1}, at least two other isometric spheres must pass through P_2, each forming a face of R. Hence P_2 is also a vertex.

Let $T_1, T_2, \cdots, T_m$ be the transformations whose isometric spheres form the faces at P_1. Then $P_k = T_k(P_1)$ is a vertex. Conversely, to each vertex congruent to P_1 by U say, there corresponds an isometric sphere Σ_U through P_1. For, P_1 on the boundary of R cannot lie inside Σ_U; if it lies outside Σ_U then $P' = U(P_1)$ lies inside Σ'_U and not on the boundary of R. The number of vertices congruent to P_1 is therefore finite.

Applying also Theorem 8 we have

THEOREM 22. *The boundary points of class* (γ) *form finite sets of congruent vertices. All vertices of a set are equidistant from the center of* Σ_0.

Points of Class (δ). A point of tangency of two isometric spheres lies necessarily on Σ_0. One or two of the tangent spheres form faces of R according as the spheres lie on the same or opposite sides of the tangent plane. Boundary points of this class do not occur for a group improperly discontinuous on Σ_0.

Angles at sets of congruent edges and vertices.

THEOREM 23. *The sum of the dihedral angles at the edges of a set of congruent edges is* 2π *or a sub-multiple of* 2π.

Let $E_1, E_2, \cdots, E_m$ be the set of edges congruent respectively by $T_1, T_2, \cdots, T_m$, with $T_m(E_m) = E_1$. Let the faces at E_1 be F_0, F_1.

The transformation $U_1 = T_mT_{m-1} \cdots T_2T_1$ has E_1 as fixed edge. Let

$$U_m = T_m;\quad U_{m-1} = T_mT_{m-1};\quad \cdots;\quad U_2 = T_mT_{m-1} \cdots T_2;$$

and

$$U_1 = T_mT_{m-1} \cdots T_2T_1.$$

Of these U_m carries E_m into E_1, the dihedral angle in R at E_m into an equal angle at E_1, and the region R into R_m joining onto R along the face F_0. U_{m-1} carries E_{m-1} into E_1, the dihedral angle about E_{m-1} into an equal one at E_1, and R into R_{m-1}, fitting onto R_m along the open face of R_m through E_1.

Finally, U_1 carries E_1 into E_1, the dihedral angle at E_1 into an equal one at E_1, and R into R_1, fitting onto R_2 along its open face through E_1.

If R_1 coincides with R, then U_1 is the identical transformation and the sum of the dihedral angles at the edges is 2π.

If $U_1 \neq 1$, we make $U_1, U_1^2, U_1^3, \cdots$ Each of these has an isometric sphere through E_1, so the number of possible repetitions of U_1 is finite. Then with some U_1^k the dihedral angle about E_1 is completely filled. The angles of the cycle have been used $k+1$ times; hence their sum is $2\pi/(k+1)$.

THEOREM 24. *The sum of the solid angles at the vertices of a set of congruent vertices is $4\pi/m$, m a positive integer.*

It is a property of conformal space transformations that solid angles are preserved.

Make the inverses of all the transformations with isometric spheres through P_1. R and the transforms of points of R near the vertices of a cycle fill out the solid angle about P_1. Each solid angle at a vertex has the same number of transforms at P_1. Thus if $T_1, \cdots, T_m$ carry P_k to P_1 and if S carries P_l to P_k, then $T_1S, \cdots, T_mS$ and no others carry P_l to P_1. Then the sum of the solid angles of the set is $4\pi/m$.

Generating transformations of the group. A set of transformations $T_1, T_2, \cdots$ is said to *generate* a group G if every transformation of G is a combination of the set.

The following theorem can be readily proved:

THEOREM 25. *The set of transformations which connect the faces of R form a set of generating transformations of the group.*

Applications to properly discontinuous groups in the plane. Let M be a properly discontinuous group of linear transformations. A sufficient condition for finding a generating set for M is that it be possible to join every ordinary point to a point of R by a curve not passing through a limit point. In such a case, the transformations connecting the sides of R form a generating set for M.

If this condition is not satisfied, we are able to find a generating set for M as follows. Project the plane stereographically on Σ_0 and form the corresponding space group G. The region R' for G is found by constructing the isometric spheres. R' must contain in its interior a point of Σ_0, for otherwise G, and hence M, would be improperly discontinuous. A generating set of transformations for G is the set connecting the faces of R'. By projecting Σ_0 back on the plane those parts of Σ_0 belonging to R' form a region R for M (possibly disconnected). The set of linear transformations corresponding to the generating set for G forms a generating set for M.

ON SUMMABILITY OF DOUBLE SEQUENCES.*

By Ralph Palmer Agnew.†

1. *Introduction.* Let $\| a_{mi} \|$ and $\| b_{nj} \|$ be two triangular matrices of real or complex constants satisfying the conditions

$$(1.1) \qquad \text{for each } i, \lim_{m\to\infty} a_{mi} = 0; \qquad \text{for each } j, \lim_{n\to\infty} b_{nj} = 0,$$

$$(1.2) \qquad \text{for each } m, \sum_{i=0}^{m} | a_{mi} | < K; \qquad \text{for each } n, \sum_{j=0}^{n} | b_{nj} | < K,$$

K being a constant independent of m and n, and

$$(1.3) \qquad \lim_{m,n\to\infty} C_{mn} = 1 \quad \text{where} \quad C_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi}b_{nj}.$$

With each convergent or divergent double sequence s_{ij}, we associate a transform S_{mn} defined by

$$F \equiv F(a, b) \qquad S_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi}b_{nj}s_{ij}.$$

A sequence s_{ij} is said to be "summable F" to S if its transform S_{mn} converges to S, to be "bounded F" if S_{mn} is uniformly bounded for all m and n, and to be "ultimately bounded F" if $\limsup_{m,n\to\infty} | S_{mn} | < \infty$.

Results of G. M. Robison ‡ show that each *bounded* convergent sequence must be summable F to the value to which it converges; but results of T. Kojima § show that conditions much more severe than (1.1), (1.2) and (1.3) are necessary to ensure that each convergent sequence shall be summable F. Hence F may be non-regular, for the simple reason that there may be unbounded convergent sequences which it fails to evaluate.

The transformation F is of the type called *factorable* by C. R. Adams, and has been investigated by C. R. Adams ¶ and F. Lösch.‖ An important special case has been considred by S. Bochner.**

* Presented to the American Mathematical Society, March 25, 1932.

† National Research Fellow.

‡ G. M. Robison, *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 50-73.

§ T. Kojima, *Tohoku Mathematical Journal*, Vol. 21 (1922), pp. 3-14.

¶ C. R. Adams, I, *Bulletin of the American Mathematical Society*, Vol. 37 (1931), pp. 741-748; II, *Transactions of the American Mathematical Society*, Vol. 34 (1932), pp. 215-230.

‖ F. Lösch, *Mathematische Zeitschrift*, Vol. 34 (1931), pp. 281-290. Lösch postulates, instead of (1.3), the two conditions $\lim_{m\to\infty} \sum_{i=0}^{m} a_{mi} = 1$, $\lim_{n\to\infty} \sum_{j=0}^{n} a_{nj} = 1$; this difference is, however, as Adams II points out, quite trivial.

** S. Bochner, *Mathematische Zeitschrift*, Vol. 35 (1932), pp. 122-126.

Lösch (*loc. cit.*, p. 282) has shown that *if s_{ij} converges to s and S_{mn} is bounded, then s_{ij} is summable F to s;* and (p. 287) that *if s_{ij} converges to s and is summable F to S, then $S = s$.* These results are of great interest since on one hand they show an extent to which unbounded convergent sequences are summable F, and on the other hand that F is consistent with convergence. The latter result has important applications in the theory of series of functions; for example, if a double trigonometric series is summable F to a function $f(x, y)$, then the series must converge to $f(x, y)$ for all values of x and y for which it converges.

It is the object of the present paper to prove and discuss the following and a related theorem.

THEOREM 1. *If s_{ij} converges to s and if there exist an index Q and two sequences α_m and β_n of constants such that*

$$\text{(1.4)} \qquad \textit{for each } m > Q, \qquad |S_{mn}| < \alpha_m, \qquad n > Q,$$

$$\text{(1.5)} \qquad \textit{for each } n > Q, \qquad |S_{mn}| < \beta_n, \qquad m > Q,$$

then s_{ij} is summable F to s.

This theorem may be stated as follows. *If a double sequence converges and if each sufficiently advanced row and column of its F-transform is bounded, then the sequence is summable F to the value to which it converges.*

2. *Consequences of Theorem* 1. Before passing to a proof of Theorem 1, we give two of its corollaries in Theorems 2 and 3.

THEOREM 2. *If s_{ij} converges to s and is ultimately bounded F, then s_{ij} is summable F to s.**

THEOREM 3. *If s_{ij} converges to s and is summable F to S, then $S = s$.*

Theorem 2 contains the first result of Lösch mentioned in § 1; Theorem 3 is, except for the fact that our transformations are slightly more general than those of Lösch, precisely the second result of Lösch.

Theorem 1 also contains the following theorem of Adams I, p. 743; if s_{ij} converges to s, and if there exist sequences $A_0, A_1, A_2, \cdots$ and $B_0, B_1, B_2, \cdots$ of constants such that for each $j \geqq 0$ and each $i \geqq 0$ and for all m and n,

$$\text{(2.1)} \qquad \left|\sum_{i=0}^{m} a_{mi}s_{ij}\right| < A_j; \qquad \left|\sum_{j=0}^{n} b_{nj}s_{ij}\right| < B_i,$$

* It follows at once from (1.2) that each bounded sequence is bounded F; hence Theorem 2 includes the result of Robison that each bounded convergent sequence is summable F to the value to which it converges.

then S_{mn} converges to S. For, suppose the hypotheses of Adams' theorem hold; then for each fixed $m \geqq 0$ we have

$$(2.2) \qquad | S_{mn} | \leqq \sum_{i=0}^{m} | a_{mi} | \, | \sum_{j=0}^{n} b_{nj} s_{ij} | \leqq \sum_{i=0}^{m} | a_{mi} | B_i.$$

An analogous set of inequalities holds for each fixed $n \geqq 0$. Thus we see that the hypotheses of Theorem 1 hold with $Q = -1$.*

3. *A Lemma.* The following lemma will be used in proofs of our theorems.

LEMMA 1. *Let R be a non-negative integer, let g_{mi} and G_{in} be double sequences of real or complex constants, and let*

$$(3.1) \qquad \lim_{m \to \infty} g_{mi} = 0, \qquad (i = 0, 1, 2, \cdots).$$

If there is an index N and a sequence H_m of constants such that whenever $n > N$ we have

$$(3.2) \qquad | g_{m0} G_{0n} + g_{m1} G_{1n} + \cdots + g_{mR} G_{Rn} | < H_m, \qquad m > N,$$

then

$$(3.3) \qquad \lim_{m,n \to \infty} (g_{m0} G_{0n} + g_{m1} G_{1n} + \cdots + g_{mR} G_{Rn}) = 0.$$

We prove this lemma by induction. It is easy to show, by considering separately the case where $g_{m0} = 0$ for all $m > N$ and the case where $g_{\mu 0} \neq 0$ for some fixed $\mu > N$, that the lemma holds when $R = 0$.

Assuming that the lemma holds when $R = 0, 1, 2, \cdots, \rho - 1$, we prove that it holds for $R = \rho$ by considering the infinite matrix

$$(3.4) \qquad \| g_{mi} \| \qquad (m = N + 1, n + 2, \cdots ; \; i = 0, 1, 2, \cdots, \rho).$$

If this matrix has rank less than $(\rho + 1)$, then the columns are linearly

* We give a few remarks bearing on the relations between our Theorem 1 and Theorem 2 of Adams II. A transformation T is said to be regular for a class $\mathcal{S}$ of sequences if each convergent sequence belonging to $\mathcal{S}$ is summable T to the value to which it converges. With this terminology, Adams shows that F is regular for the class $\mathcal{B}$ of all sequences which are bounded F. Now our Theorem 1 shows that F is regular for the class $\mathcal{L}$ of all sequences having F-transforms of which each sufficiently advanced row and column is bounded.

It is clear that $\mathcal{L}$ contains all sequences which are summable F; hence $\mathcal{L}$ is, apart from divergent sequences, the largest class of sequences for which F is regular. It is also clear that in case F has an inverse, and in many other cases as well, a convergent sequence may belong to $\mathcal{L}$ and fail to belong to $\mathcal{B}$. Hence Theorem 2 of Adams II, and the later Theorems of Adams II which depend upon it, can be made stronger as well as easier to apply by using the class $\mathcal{L}$ instead of the class $\mathcal{B}$.

dependent, i. e. there exists a system $\lambda_0, \lambda_1, \cdots, \lambda_\rho$ of constants (not all zero) such that

$$\lambda_0 g_{m0} + \lambda_1 g_{m1} + \cdots + \lambda_\rho g_{m\rho} = 0, \qquad m > N.$$

Selecting an index α such that $\lambda_\alpha \neq 0$, we obtain

$$g_{m\alpha} = (\lambda_0 g_{m0} + \cdots + \lambda_{\alpha-1} g_{m,\alpha-1} + \lambda_{\alpha+1} g_{m,\alpha+1} + \cdots + \lambda_R g_{mR})/\lambda_\alpha.$$

When we substitute for $g_{m\alpha}$ in the relations obtained by setting $R = \rho$ in (3.2) and (3.3), we find that we have reduced our problem to the case $R < \rho$.

If on the other hand the matrix (3.4) has rank $(\rho + 1)$, let

$$\det(g_{m_\alpha i}) \qquad (\alpha = 0, 1, \cdots, \rho;\ i = 0, 1, \cdots, \rho) \tag{3.5}$$

be a non-vanishing $(\rho + 1)$-rowed determinant selected from its elements. Let us now consider only values of n which exceed N. Since $m_0, m_1, \cdots, m_\rho$ all exceed N, we have by (3.2)

$$| g_{m_\alpha 0} G_{0n} + g_{m_\alpha 1} G_{1n} + \cdots + g_{m_\alpha \rho} G_{\rho n} | < H_{m_\alpha}, \quad (\alpha = 0, 1, \cdots, \rho). \tag{3.6}$$

Let us set

$$g_{m_\alpha 0} G_{0n} + g_{m_\alpha 1} G_{1n} + \cdots + g_{m_\alpha \rho} G_{\rho n} = H_{m_\alpha n} \quad (\alpha = 0, 1, \cdots, \rho). \tag{3.61}$$

Then by (3.6) we have for all values of n under consideration

$$| H_{m_\alpha n} | < H_{m_\alpha}, \qquad (\alpha = 0, 1, \cdots, \rho). \tag{3.7}$$

Since (3.5) does not vanish, we can solve the equations (3.61) for G_{in} obtaining

$$G_{in} = \Delta_{i0} H_{m_0 n} + \Delta_{i1} H_{m_1 n} + \cdots + \Delta_{i\rho} H_{m_\rho n}, \quad (i = 0, 1, \cdots, \rho). \tag{3.8}$$

where the Δ_{ij} are constants depending only on the elements of the determinant (3.5). It follows from (3.7) and (3.8) that each of the sequences

$$G_{0n}, G_{1n}, \cdots, G_{\rho n} \tag{3.9}$$

is bounded for all $n > N$; hence we may use (3.1) to obtain (3.3) for $R = \rho$, and the proof by induction is complete.

4. *Proof of Theorem* 1. Let s_{ij} be any given sequence converging to s and having an F-transform satisfying the hypotheses of Theorem 1. Given $\epsilon > 0$, choose an index R which is greater than Q and also so great that

$$| s_{mn} - s | < \epsilon/2K^2, \qquad m, n > R, \tag{4.1}$$

K being the constant in (1.2). When $m, n > R$, we have

$$\begin{aligned} S_{mn} - s &= \{\sum_{i=0}^{R} \sum_{j=0}^{n} + \sum_{i=0}^{m} \sum_{j=0}^{R} - \sum_{i=0}^{R} \sum_{j=0}^{R} + \sum_{i=R+1}^{m} \sum_{j=R+1}^{n}\} a_{mi} b_{nj} (s_{ij} - s) + s(C_{mn} - 1) \\ &= S_{mn}{}^{(1)} + S_{mn}{}^{(2)} - S_{mn}{}^{(3)} + S_{mn}{}^{(4)} + s(C_{mn} - 1). \end{aligned}$$

Using (4.1) and (1.2), we see that $| S_{mn}^{(4)} | < \epsilon/2$. Also (1.1) and (1.3) imply that we can choose $r > R$ so great that $| -S_{mn}^{(3)} + s(C_{mn} - 1) | < \epsilon/2$ when $m, n > r$. Then on one hand

(4.2) $$| S_{mn} - s | < | S_{mn}^{(1)} | + | S_{mn}^{(2)} | + \epsilon, \qquad m, n > r$$

and on the other hand

(4.3) $$| S_{mn}^{(1)} + S_{mn}^{(2)} | < | S_{mn} | + | s | + \epsilon, \qquad m, n > r.$$

We proceed to show that

(4.4) $$\lim_{m,n\to\infty} S_{mn}^{(1)} = 0.$$

Using (1.1), we see that for each fixed m, $\lim_{n\to\infty} S_{mn}^{(2)} = 0$; hence there is a sequence H'_m of constants such that for each m,

(4.5) $$| S_{mn}^{(2)} | < H'_m, \qquad n > r.$$

Combining (1.4), (4.3), and (4.5) we obtain for each $m > r$

(4.6) $$| S_{mn}^{(1)} | < H_m \qquad n > r$$

where

(4.7) $$H_m = H'_m + \alpha_m + | s | + \epsilon.$$

Introducing the notation

(4.8) $$A_{in} = \sum_{j=0}^{n} b_{nj}(s_{ij} - s),$$

in (4.6), we obtain for each fixed $m > r$

(4.9) $$| S_{mn}^{(1)} | = | a_{m0}A_{0n} + a_{m1}A_{1n} + \cdots + a_{mR}A_{Rn} | < H_m, \qquad n > r.$$

An application of Lemma 1 yields (4.4). An analogous argument shows that $\lim_{m,n\to\infty} S_{mn}^{(2)} = 0$. It therefore follows from (4.2) that $\lim_{m,n\to\infty} S_{mn} = s$ and Theorem 1 is proved.

5. *A variation of Theorem* 1. Lösch, *loc. cit.*, pp. 285-287, imposes upon the elements of $\| a_{mi} \|$ and $\| b_{nj} \|$ the following condition. Corresponding to each pair q and Q of indices, there exist two systems $m_0 < m_1 < \cdots < m_q$ and $n_0 < n_1 < \cdots < n_q$ of indices such that $m_0 > Q$, $n_0 > Q$, and each of the determinants

(5.1) $$\det(a_{m_j i}); \quad \det(b_{n_i j}) \qquad (i = 0, 1, \cdots, q;\ j = 0, 1, \cdots, q)$$

does not vanish. Lösch shows that when this condition, which we shall designate by (5.1), as well as (1.1), (1.2), and (1.3) hold, then each convergent sequence summable F must be also bounded F. We now give a theorem which includes this result.

THEOREM 4. *Let F satisfy* (5.1) *as well as* (1.1), (1.2), *and* (1.3). *If s_{ij} converges and has a transform satisfying the hypotheses of Theorem* 1, *then there exist an index r and two sequences α'_m and β'_n of constants such that*

$$\text{(5.2)} \qquad \text{for each } m, \ |S_{mn}| < \alpha'_m, \qquad n > r,$$

and

$$\text{(5.3)} \qquad \text{for each } n, \ |S_{mn}| < \beta'_n, \qquad m > r.$$

This theorem may be stated as follows. *Let F satisfy* (5.1) *as well as* (1.1), (1.2), *and* (1.3). *If s_{ij} converges and if each sufficiently advanced row and column of its F-transform is bounded, then each row and column of its F-transform is bounded.* It follows that, when F satisfies these conditions, *each convergent sequence which is ultimately bounded F is also bounded F.*

To prove Theorem 4, we proceed precisely as in the proof of Theorem 1 to obtain (4.9). The hypothesis (5.1) ensures the existence of a system $R < m_0 < m_1 < \cdots < m_R$ of indices such that the determinant

$$\text{(5.4)} \qquad \det(a_{m_j i}) \qquad (i = 0, 1, \cdots, R;\ j = 0, 1, \cdots, R)$$

does not vanish. Using the fact that (4.9) holds when $m = m_0, m_1, \cdots, m_R$, we see as in the latter part of the proof of Lemma 1 that each of the sequences

$$\text{(5.5)} \qquad A_{0n}, A_{1n}, \cdots, A_{Rn}$$

is bounded for all $n > r$. Since this is a finite set of sequences they are uniformly bounded for all $n > r$, i. e. there is a constant A_R such that whenever $i \leqq R$, we have

$$\text{(5.6)} \qquad |A_{in}| < A_R \qquad n > r.$$

Hence when $m \leqq R$ we have

$$\text{(5.7)} \qquad |S_{mn}| \leqq \sum_{i=0}^{m} |a_{mi}|\ |A_{in}| < KA_R \qquad n > r.$$

But R was chosen greater than Q, and r greater than R; hence it results from (1.4) that when $m > R$,

$$\text{(5.8)} \qquad |S_{mn}| < \alpha_m \qquad n > r.$$

Letting $\alpha'_m = KA_R$ when $m \leqq R$ and $\alpha'_m = \alpha_m$ when $m > R$, we see that (5.2) follows from (5.7) and (5.8). An analogous argument yields (5.3) and Theorem 4 is proved.

6. *Applications.* Let V and W represent any methods which associate with each double sequence a transformed double sequence, and suppose

$$\text{(6.1)} \qquad V = F(a, b)W$$

where F is defined as in § 1 and FW represents the transformation which associates with a sequence the F-transform of its W-transform. An application of Theorem 1 gives the following result. *If* $V = FW$, *and if* s_{ij} *is summable* W *to* s *and each sufficiently advanced row and column of its* V*-transform is bounded, then* s_{ij} *is summable* V *to* s. A corollary of this result gives an application of Theorem 2, namely, *if* $V = FW$ *and if* s_{ij} *is summable* W *to* s *and ultimately bounded* V, *then* s_{ij} *is summable* V *to* s. A further corollary gives an application of Theorem 3, namely, *if* $V = FW$, *and* s_{ij} *is summable* W *to* s *and summable* V *to* S, *then* $S = s$; in other words *V and W are consistent.*

There is an important class of transformations V and W for which (6.1) holds. Let

$$A^{(\alpha)}:\quad A_n^{(\alpha)} = \sum_{k=0}^{n} a_{nk}^{(\alpha)} s_k;\qquad B^{(\alpha)}:\quad B_n^{(\alpha)} = \sum_{k=0}^{n} b_{nk}^{(\alpha)} s_k\qquad (\alpha = 1, 2)$$

be four simple-sequence transformations about which nothing is assumed other than that $A^{(1)} = C^{(1)}B^{(1)}$ and $A^{(2)} = C^{(2)}B^{(2)}$ where $C^{(1)}$ and $C^{(2)}$ are regular transformations with triangular matrices $\| c_{nk}^{(1)} \|$ and $\| c_{nk}^{(2)} \|$. It is easy to show that

$$(6.2)\qquad A^{(1)} \odot A^{(2)} = (C^{(1)} \odot C^{(2)})\,(B^{(1)} \odot B^{(2)})$$

where $A^{(1)} \odot A^{(2)}$ is the double-sequence transformation defined by *

$$A_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi}^{(1)} a_{nj}^{(2)} s_{ij}$$

and $B^{(1)} \odot B^{(2)}$ and $C^{(1)} \odot C^{(2)}$ are similarly defined. The transformations $A^{(1)} \odot A^{(2)}$ and $B^{(1)} \odot B^{(2)}$ need not have the form F since the conditions analogous to (1.1), (1.2) and (1.3) may fail to hold. However regularity of $C^{(1)}$ and $C^{(2)}$ ensures that $C^{(1)} \odot C^{(2)}$ is of the form F; hence (6.2) is of the form (6.1) and our results may be applied. We state, for reference, the following theorems.

THEOREM 5. *Let* $A^{(1)} = C^{(1)}B^{(1)}$ *and* $A^{(2)} = C^{(2)}B^{(2)}$ *where* $C^{(1)}$ *and* $C^{(2)}$ *are regular. If* s_{ij} *is summable* $B^{(1)} \odot B^{(2)}$ *to* s, *and if each sufficiently advanced row and column of the* $A^{(1)} \odot A^{(2)}$ *transform of* s_{ij} *is bounded, then* s_{ij} *is summable* $A^{(1)} \odot A^{(2)}$ *to* s.

THEOREM 6. *If* $A^{(1)} = C^{(1)}B^{(1)}$ *and* $A^{(2)} = C^{(2)}B^{(2)}$ *where* $C^{(1)}$ *and* $C^{(2)}$ *are regular, then* $A^{(1)} \odot A^{(2)}$ *and* $B^{(1)} \odot B^{(2)}$ *are consistent.*

7. *The unsymmetric case.* Let I represent the identity transformation. Theorem 3 shows that if A and B are regular, then $A \odot B$ and $I \odot I$ are

* This is the notation of Adams I and II.

consistent. We now propose to prove and give consequences of the following theorem.

THEOREM 7. *If A and B are regular, then* $A \odot I$ and $I \odot B$ *are consistent.*

The proof of Theorem 7 follows at once from Theorem 5 and the following lemma.

LEMMA 2. *Let A and B be transformations satisfying* (1.2). *If each sufficiently advanced row of the $A \odot I$ transform of s_{ij} is bounded and each sufficiently advanced column of the $I \odot B$ transform of s_{ij} is bounded, then each sufficiently advanced row and column of the $A \odot B$ transform of s_{ij} is bounded.*

To prove this lemma, let s_{ij} be a sequence satisfying its hypotheses, and choose sequences α_m and β_n of constants and an index Q such that for each $m > Q$

$$(7.1) \qquad \left| \sum_{i=0}^{m} a_{mi} s_{in} \right| < \alpha_m \qquad n > Q$$

and for each $n > Q$

$$(7.2) \qquad \left| \sum_{j=0}^{n} b_{nj} s_{mj} \right| < \beta_n \qquad m > Q.$$

Letting S_{mn} represent the $A \odot B$ transform of s_{ij}, we may write for each fixed $n > Q$

$$S_{mn} = \sum_{i=0}^{Q} \sum_{j=0}^{n} a_{mi} b_{nj} s_{ij} + \sum_{i=Q+1}^{m} a_{mi} \sum_{j=0}^{n} b_{nj} s_{ij}.$$

Using (1.2) and (7.2), we obtain

$$S_{mn} \leqq K^2 \max_{0 \leqq i \leqq Q;\ 0 \leqq j \leqq n} | s_{ij} | + K\beta_n \qquad m > Q.$$

The right member of this inequality depends only on n; hence each column of S_{mn} with a fixed index $n > Q$ is bounded. An analogous argument shows that each row of S_{mn} with a fixed index $m > Q$ is bounded and the lemma is proved.

A transformation A is said to include a transformation B (written $A \supset B$) if each sequence summable B is also summable A to the same value. From Theorem 7, we obtain the following more inclusive result.

THEOREM 8. *If $A^{(1)} \supset B^{(1)}$ and $B^{(2)} \supset A^{(2)}$, and $B^{(1)}$ and $A^{(2)}$ have inverses, then $A^{(1)} \odot A^{(2)}$ and $B^{(1)} \odot B^{(2)}$ are consistent.*

Since $A^{(1)}[B^{(1)}]^{-1}$ and $B^{(2)}[A^{(2)}]^{-1}$ are regular it follows from Theorem 7 that $A^{(1)}[B^{(1)}]^{-1} \odot I$ and $I \odot B^{(2)}[A^{(2)}]^{-1}$ are consistent; hence

$$A^{(1)} \odot A^{(2)} = \{A^{(1)}[B^{(1)}]^{-1} \odot I\}\{B^{(1)} \odot A^{(2)}\}$$

and

$$B^{(1)} \odot B^{(2)} = \{I \odot B^{(2)}[A^{(2)}]^{-1}\}\{B^{(1)} \odot A^{(2)}\}$$

are consistent and Theorem 8 is proved.

8. *Conclusion.* Combining Theorems 6 and 8 we obtain

THEOREM 9. *If $A^{(1)}$ includes or is included by $B^{(1)}$, if $A^{(2)}$ includes or is included by $B^{(2)}$, and if $A^{(1)}$, $B^{(1)}$, $A^{(2)}$, and $B^{(2)}$ all have inverses, then $A^{(1)} \odot A^{(2)}$ and $B^{(1)} \odot B^{(2)}$ are consistent.*

Theorems 5, 6, 8, and 9 have many immediate applications. We mention only a few of the applications of Theorem 9 to Cesàro and Holder methods of summability.

Let $C(r)$ and $H(r)$ denote respectively the simple-sequence Cesàro and Holder transformations of order r; then the double-sequence Cesàro and Holder transformations $C(r_1, r_2)$ and $H(r_1, r_2)$ are defined to be $C(r_1) \odot C(r_2)$ and $H(r_1) \odot H(r_2)$ respectively. Applying Theorem 9, we see that *if r_1, r_2, r_3, and r_4 are all real and greater than -1, then any two of the transformations $C(r_1, r_2)$, $C(r_3, r_4)$, $H(r_1, r_2)$, and $H(r_3, r_4)$ are consistent.** To this set of consistent methods can be added methods of the form $C(r_1) \odot H(r_2)$ and $H(r_1) \odot C(r_2)$.

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* This application of Theorem 9 depends upon the fact that when r and r' are real and greater than -1, we have at least one of the relations $C(r) \supset C(r')$ and $C(r') \supset C(r)$, and the fact that $C(r)$ and $H(r)$ are equivalent when $r > -1$. For references to literature, see E. Kogbetliantz, *Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques*, Paris (1931), pp. 17-19. It should be noted that we cannot establish a part of our consistency theorem by showing that $C(r', \rho') \supset C(r, \rho)$ when $r' > r > -1$ and $\rho' > \rho > -1$; that the latter result does not hold follows from the fact that $C(1, 1)$ and $C(0, 0)$ are overlapping methods of summability in the sense that each evaluates certain sequences which the other fails to evaluate. It should be noted also that the equivalence theorem for simple-sequence Cesàro and Holder methods cannot be extended to double-sequence transformations. In fact we can show that $C(2, 2)$ and $H(2, 2)$ are overlapping methods of summability by using the two identities

$$C(2,2) = \{C(2)[H(2)]^{-1} \odot C(2)[H(2)]^{-1}\} H(2,2)$$

and

$$H(2,2) = \{H(2)[C(2)]^{-1} \odot H(2)[C(2)]^{-1}\} C(2,2)$$

and the Kojima (*loc. cit.*) conditions for regularity of double sequence transformations.

ON THE EXISTENCE OF CRITICAL POINTS OF GREEN'S FUNCTIONS FOR THREE-DIMENSIONAL REGIONS.

By Tsai-Han Kiang.

1. *Introduction.* The regions we shall consider here are the connected, closed, and bounded three-dimensional regions, for which the Dirichlet problem is possible and which are 3-complexes * in the sense of analysis situs. For simplicity let us call such a region an *admissible region.* Of an admissible region the connectivity numbers R_0 and R_3 are obviously 1 and 0 respectively. We shall investigate the critical points † of Green's functions first for general admissible regions with R_1 and R_2 not both zero (Theorem 1), then for admissible regions with R_1 and R_2 both zero but not homeomorphic with a spherical region (Theorem 2), and finally for admissible regions not only with R_1 and R_2 both zero but also homeomorphic with a spherical region (Theorem 4).‡

2. *Critical points of Green's functions for general admissible regions.* For our present purpose we shall state a special case of a theorem in a previous paper by the author § as the following theorem:

Theorem A. *Suppose a non-degenerate* ¶ *function $f(x, y, z)$ and its region R of definition, a finite closed admissible region, fulfill the following conditions:*

(1) *The function f is harmonic* ‖ *in a region containing R in its interior.*

* The terminology of analysis situs will be used in the sense as defined by J. W. Alexander in his paper "Combinatorial Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 301-329. But the term "cycle" will be used in place of his term "closed chain" and the symbol "R_i" in place of his symbol "P_i" for the i-th connectivity number.

† A point (x, y, z) is called a *critical point* of a function $f(x, y, z)$ if all the three partial derivatives of first order of f vanish at the point.

‡ See J. J. Gergen, "Mapping of a General Type of Three-Dimensional Region on a Sphere," *American Journal of Mathematics*, Vol. 52 (1930), pp. 197-198. He has proved from different considerations a particular case of our Theorem 1 and a part of our Theorem 4a for a different region.

§ Tsai-Han Kiang, "On the Critical Points of Non-degenerate Newtonian Potentials," Theorem A, *American Journal of Mathematics*, Vol. 54 (1932), pp. 92-109.

¶ A critical point of f is said to be *degenerate* or *non-degenerate* according as the hessian of f vanishes at the point or not. The function f is said to be a *degenerate* or *non-degenerate function* in a region according as it has or has no degenerate critical point in the region.

‖ A function f is said to be *harmonic at a point* if its partial derivatives of second order are continuous and satisfy Laplace's differential equation throughout some neigh-

(2) *The boundary B of R consists of two sets B' and B'' of closed non-singular analytic equipotential surfaces* * *of f, such that the value c' of f on B' is greater than the value c'' of f on B''.*†

Let M_k $(k = 1, 2)$ be the number of critical points of the k-th type ‡ *of f in R. Let R_i and R'_i $(i = 0, 1, 2)$ be the i-th connectivity numbers of the complexes R and B'' respectively. Then there exist non-negative integers M_k^+ and M_k^- such that*

$$M_k = M_k^+ + M_k^-, \qquad (k = 1, 2);$$
$$R_0 - R'_0 = -M_1^-, \quad R_1 - R'_1 = M_1^+ - M_2^-, \quad R_2 - R'_2 = M_2^+.$$

We need also the following two lemmas proved by Kellogg.§

LEMMA A. *Suppose $g(x, y, z)$ is the Green's function for a three-dimensional region D with the pole at an interior point of D. Let c' be a non-critical value* ¶ *of g in D. Then the points of D satisfying $g = c'$ constitute one or more closed non-singular analytic surfaces in D, bounding a non-singular analytic region* ‖ *of points of D satisfying $g \geqq c'$.*

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borhood of that point. It is said to be *harmonic in a region* if it is continuous in the region and harmonic at all interior points of the region.

* A *regular surface element* and a *regular surface* will be used as defined by Kellogg in his book, *Foundations of Potential Theory*, Berlin (1929). A regular surface element will be said to be *analytic*, if it admits for some orientation of coördinate axes a representation $z = F(x, y)$ where F is analytic. A closed regular surface will be said to be *non-singular* (*non-singular analytic*), if every point of the surface is an interior point of a regular (regular analytic) surface element.

† This condition implies that the normal derivative of f never vanishes on B, and as a consequence of harmonicity of f that the constant c' is greater but the constant c'' is less than the value of f at any interior point of R.

‡ Suppose (x^0, y^0, z^0) be a non-degenerate critical point of f. Let f^0_{xx}, f^0_{xy}, etc. be the partial derivatives of second order of f evaluated at the critical point. By a real non-singular transformation linear in x, y, z, the non-singular quadratic form

$$f^0_{xx}(x - x^0)^2 + f^0_{yy}(y - y^0)^2 + f^0_{zz}(z - z^0)^2 + 2f^0_{xy}(x - x^0)(y - y^0)$$
$$+ 2f^0_{yz}(y - y^0)(z - z^0) + 2f^0_{zx}(z - z^0)(x - x^0)$$

can be reduced to one of the forms:

$$\pm X^2 \quad \pm Y^2 \quad \pm Z^2.$$

If the number of negative signs of the reduced form is i, the critical point is said to be of i-th type.

The critical points of 0-th type and third type are critical points at which f has minimum and maximum values respectively. Since f is harmonic in R and since its normal derivative never vanishes on B, $M_0 = M_3 = 0$.

§ Kellogg, *loc. cit.*, pp. 238-239 and p. 276. For Lemma B note also the proposition (c) on the next page.

¶ The value of g at a point of D will be called a *critical* value or a *non-critical value* of g according as the point is or is not a critical point of g.

‖ A *non-singular analytic region* is a finite region in three-dimensional space bounded by one or more closed non-singular analytic surfaces.

LEMMA B. *In a closed region entirely in the interior of D the Green's function has only a finite number of critical values.*

Now let D be an admissible region with the connectivity numbers $R_0 = 1$, R_1, R_2, $R_3 = 0$, and $g(x, y, z)$ the Green's function for D with the pole at an interior point P of D. From the very definition of R_1 and R_2 there exist in D a set of R_1 1-cycles linearly independent with respect to bounding and a set of R_2 2-cycles linearly independent with respect to bounding. We may assume that these R_1 and R_2 cycles are in the interior of D and do not contain the pole P of the Green's function g. Because g is positive in the interior of D, the values of g on these cycles have a positive lower bound, d say. From Lemma B there is a non-critical value c' of g such that $d/2 > c' > d/3$. Let the region of points of D satisfying $g \geqq c'$ be denoted by N. From Lemma A the region N is a non-singular analytic region bounded by one or more closed non-singular analytic equipotential surfaces B' represented by $g = c'$, lies in the interior of D, and contains the cycles in its interior. Moreover, the region N is a 3-complex.* From the definition of connectivity numbers again, the connectivity numbers of N are 1, $R_1 + a_1$, $R_2 + a_2$, 0, where a_1 and a_2 are non-negative integers.

As in the proof of Lemma 4 in the previous paper by the author, *loc. cit.*, the following facts can be established: (a) For a sufficiently large positive constant c'', the points of N satisfying $g = c''$ constitute a single non-singular analytic equipotential surface B'' enclosing the pole P of g. (b) The bounded closed region bounded by B'' is homeomorphic to a closed 3-cell. (c) In this region g has no critical point.

From (b) above, B'' is homeomorphic with a 2-sphere and hence its connectivity numbers are 1, 0, 1, 0. Let R denote the region obtained from N with the interior of B'' removed. From the connectivity numbers of N, the connectivity numbers of R are evidently 1, $R_1 + a_1$, $R_2 + a_2 + 1$, 0, where a_1 and a_2 are non-negative integers.

In case g is degenerate in D, by definition g has at least one critical point in D. Suppose now g is non-degenerate in the interior of the region D and hence in R. Theorem A can be most conveniently applied to the function $-g$ in R. The conclusion of Theorem A states that, when the numbers of critical points of $-g$ of k-th $(k = 1, 2)$ types in R are M_k, there exist non-negative integers M_k^+ and M_k^- such that

$$M_k = M_k^+ + M_k^-, \quad -M_1^- = 0, \quad M_1^+ - M_2^- = R_1 + a_1, \quad M_2^+ = R_2 + a_2.$$

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* S. S. Cairns, "The Celluar Structure and Approximations of Regular Spreads," *Proceedings of the National Academy of Sciences, U. S. A.*, Vol. 16 (1930), pp. 488-491.

Hence for $-g$ we have

$$M_1 = R_1 + a_1 + M_2^-, \qquad M_2 = R_2 + a_2 + M_2^-.$$

Now it is obvious from the definition of the types of critical points, that a critical point of type 1 or 2 of $-g$ is a critical point of type 2 or 1 of $+g$. Hence in R the function g has at least R_2 critical points of type 1 and R_1 critical points of type 2. The results will be summarized in the following theorem.

THEOREM 1. *Suppose the connectivity numbers R_1 and R_2 of an admissible region are not both zero. Then the Green's function for the region with the pole at an interior point has at least one critical point in the interior of the region.*

If, moreover, the Green's function is non-degenerate in the interior of the region, it has at least R_2 critical points of type 1 and R_1 critical points of type 2 in the interior of the region.

3. *Critical points of Green's functions for admissible regions with the same connectivity numbers as a spherical region but not homeomorphic with it.* Let us consider in the space the simple closed curve Q_0 with continuously turning tangent:

$$y^2 = x^4(1 - x^2), \qquad 0 \leqq x, \qquad z = 0;$$

and the point $(r, 0, 0)$ where r is a sufficiently small positive number, less than $1/3$ say. By revolving about the y-axis through two straight angles the point $(r, 0, 0)$ generates a circle C and the curve Q_0 a locus B'_0 of revolution. The origin lies on B'_0 and will be called a *singular point* of B'_0. The finite closed region T'_0 bounded by B'_0 will be called a singular region. In the interior of T'_0 is the circle C.

We can decompose T'_0 into a 3-complex with C as a 1-subcycle. Through any point of B'_0 there is a sphere tangent to B'_0 and containing no point of $T'_0 - B'_0$. From Poincare's criterion for the Dirichlet problem * there exists a Green's function for the region with the pole at any interior point. Hence T'_0 is an admissible region. Obviously the 3-complex T'_0 has the same connectivity numbers as an ordinary spherical region but is not homeomorphic with it, and the circle C does not bound any 2-complex entirely in the interior of T'_0.

Given any positive integer R_1 we can construct in the following manner an admissible region T_0 with the boundary B_0, which has the same connectivity numbers as an ordinary spherical region but is not homeomorphic with it,

* Kellogg, *loc. cit.*, p. 329.

and which has R_1 circles in $T_0 - B_0$ and linearly independent with respect to bounding in $T_0 - B_0$. Let us start with the singular region T'_0. The curve Q_0 has two horizontal tangents at the two points whose x-coördinates are equal to $(2/3)^{1/2}$. Let us subject the whole space to a reflection in the plane $x = (2/3)^{1/2}$. The singular region obtained from T'_0 and its image, each point being counted only once, has a boundary with two singular points, the origin and its image. If we subject the space to another reflection in the plane through one of the two singular points and perpendicular to the x-axis, we shall get a singular region whose boundary has three singular points. After suitable numbers of reflections of these two kinds we shall get a closed singular region T_0 whose boundary B_0 has R_1 singular points. By the same reflections, from C we get R_1 circles $C_1, C_2, \cdots, C_{R_1}$ in $T_0 - B_0$ and linearly independent with respect to bounding in $T_0 - B_0$. The region T_0 has therefore the desired properties.

The region T_0 is an example of the kind of region D in the following theorem.

THEOREM 2. *Suppose D is an admissible region with the same connectivity numbers as an ordinary spherical region but not homeomorphic with it. Suppose there are R_1 1-cycles of D, which are in the interior (in the sense of point sets) of D and are linearly independent with respect to bounding in the interior of D.*

If $R_1 \neq 0$, then the conclusions of Theorem 1 (put $R_2 = 0$) hold for the Green's function for D with the pole at an interior point P of D.

Proof. From our hypothesis there exist R_1 1-cycles in the interior of D. We may assume that these cycles do not contain the pole P. Let d be a positive lower bound of the values of the Green's function g on these cycles. There is a non-critical value c' of g such that $d/2 > c' > d/3$ (Lemma B). The connected non-singular analytic region N of points satisfying $g \geqq c'$ contains the pole P and the cycles $C_1, C_2, \cdots, C_{R_1}$ in its interior (Lemma A). Since N is a sub-region of the interior of D and since the R_1 cycles are linearly independent with respect to bounding in the interior of D, the R_1 cycles are linearly independent with respect to bounding in N. Hence the first connectivity number of the 3-complex N is at least R_1. On applying Theorem 1 to the Green's function $g - c'$ for N with the pole P, we obtain our theorem.

The following corollary is then obvious.

COROLLARY. *The Green's function for the region T_0 with the pole at an interior point has at least one critical point in the interior of T_0.*

If, moreover, the Green's function is non-degenerate in the interior of T_0, it has at least R_1 critical points of type 2 in the interior of T_0.

4. *The region T_0 as the limit of a sequence $\{T_i\}$ of non-singular regions homeomorphic with a spherical region.* Let us return to the curve Q_0 of § 3. Suppose $\{a_i\}$ $(i = 1, 2, 3, \cdots)$ is a monotonically decreasing sequence of positive numbers, whose limit is zero and whose first number a_1 is sufficiently small, less than $1/3$ say. Let us form by means of this sequence of numbers the following sequence of functions of class C':

$$F_i(x) = \begin{cases} (2x + a_i)(x - a_i)^2, & 0 \leqq x \leqq a_i, \\ 0, & a_i \leqq x; \end{cases}$$

and replace the upper branch

$$y = f(x) = x^2(1 - x^2)^{1/2}, \quad 0 \leqq x, \qquad z = 0,$$

of the curve Q_0 by the curve with continuously turning tangent:

$$y = f(x) + F_i(x), \quad 0 \leqq x, \qquad z = 0.$$

The curve Q'_i thus obtained from the whole of Q_0 is a simple open curve with continuously turning tangent, whose two endpoints are at $(0, 0, 0)$ and $(0, a_i^3, 0)$ and which has distinct horizontal tangents at the endpoints. The locus B'_i of revolution generated by revolving Q'_i about the y-axis through two straight angles is thus a non-singular surface. Let T'_i denote the finite closed non-singular region bounded by B'_i.

Let us subject the whole space to the reflections of § 3. Then as we obtained the region T_0 and its boundary B_0 from T'_0 and B'_0 respectively, we shall obtain from the region T'_i and its boundary B'_i a finite region T_i and its boundary B_i. The regions T_i are obviously admissible regions homeomorphic with an ordinary spherical region.

For $i, m = 1, 2, 3, \cdots$, we have

$$\begin{aligned} f(x) &< f(x) + F_{i+m}(x), && 0 \leqq x < a_{i+m}, \\ f(x) + F_{i+m}(x) &< f(x) + F_i(x), && 0 \leqq x < a_i. \end{aligned}$$

Hence T_{i+1} contains T_i as a sub-region and T_i contains T_0 as a sub-region. As i increases the surface B_i shrinks down to B_0. Any point not belonging to T_0 is not a point of T_i either for all values of i or for sufficiently large values of i. It is in this sense that T_i and B_i will be said to *converge from the exterior* to T_0 and B_0 respectively.

Let us observe the following two important properties of the convergent sequence $\{T_i\}$ and the limit region T_0:

(1) Any point of B_0 is a point of B_i either for all values of i or for

sufficiently large values of i. In particular, the singular points of B_0 are points of B_i for all values of i.

(2) Through any point of B_0 there is a sphere which is tangent to B_i and which contains no point of $T_i - B_i$ either for all values of i or for sufficiently large values of i. In particular, through any singular point of B_0 there is a sphere, which is tangent to B_i and contains no point of $T_i - B_i$ for all values of i.

5. *A sequence of harmonic functions* $\{h_i\}$ *in* T_0. The region T_0 is a sub-region of every region T_i $(i = 1, 2, 3, \cdots)$ and an interior point P of T_0 is an interior point of T_i. Since the Dirichlet problem is possible for T_k $(k = 0, 1, 2, \cdots)$, there exists a unique Green's function g_k for T_k with the pole at P. The function g_k is of the form

$$g_k(x, y, z) = 1/r + h_k(x, y, z),$$

where r is the distance from P to the variable point (x, y, z) of T_k, h_k is harmonic in T_k, and $h_k(x, y, z) = -1/r$ for any point (x, y, z) on the boundary B_k of T_k. As a well-known property of harmonic functions, h_k attains its maximum and minimum values only on the boundary B_k. Hence all the functions h_k are bounded in T_0, namely, by the absolute maximum and minimum values of the function $-1/r$ in the finite closed region bounded by B_0 and B_1.

Let us denote any points of T_0 and B_0 by P_0 and b_0 respectively. Since a Green's function is positive in the interior of its region, we have

$$g_0(b_0) \leqq g_{i+1}(b_0) \leqq g_i(b_0), \qquad (i = 1, 2, 3, \cdots),$$

and consequently

$$h_0(b_0) \leqq h_{i+1}(b_0) \leqq h_i(b_0), \qquad (i = 1, 2, 3, \cdots).$$

From the property of harmonic functions stated above, we infer from these inequalities the following:

$$h_0(P_0) \leqq h_{i+1}(P_0) \leqq h_i(P_0), \qquad (i = 1, 2, 3, \cdots). \tag{1}$$

Now let us consider the sequence $\{h_i\}$ of harmonic functions defined in T_0. From the relations (1), the sequence converges at any point P_0 of T_0. The limit of the sequence is thus a function $H(x, y, z)$ in T_0. By Harnack's second convergence theorem,* the sequence converges uniformly in any closed region in the interior of T_0 and the limit function H is harmonic in the interior of T_0. Moreover, from the property (1) in § 4 the value of H at any point of B_0 is equal to the value of $-1/r$ at that point.

* Kellogg, *loc. cit.*, p. 263.

Identification of H with h_0. To prove that the function H is identically equal to the function h_0, it is only necessary to prove that H takes on the continuous boundary values $-1/r$ on B_0. For, if H takes on these continuous boundary values, H is the solution of the Dirichlet problem for the region T_0 and for this boundary condition, and by the uniqueness theorem of the problem H is identically equal to h_0.

From the property (2) in § 4, through any point b_0 of B_0 there is a sphere which is tangent to B_0 and B_i and which contains no point of $T_0 - B_0$ and $T_i - B_i$ either for all values of i or for sufficiently large values of i. Let $U(A, b_0)$ be the function harmonic in the exterior of the sphere and taking on the continuous boundary values $-1/r$ on the sphere, where A denotes a variable point not in the interior of the sphere and r the distance from P to A. As we obtained (1), so for any point P_0 of T_0 we obtain the following inequalities,

$$h_0(P_0) \leqq h_i(P_0) \leqq U(P_0, b_0),$$

either for all values of i or for sufficiently large values of i. On taking the limit as i increases indefinitely, we find

$$h_0(P_0) \leqq H(P_0) \leqq U(P_0, b_0).$$

Since

$$U(b_0, b_0) = h_0(b_0) = -1/\overline{Pb_0},$$

$\overline{Pb_0}$ being the distance between P and b_0, and since both $h_0(P_0)$ and $U(P_0, b_0)$ are continuous at $P_0 = b_0$, the function $H(P_0)$ in T_0 is continuous at $P_0 = b_0$. Now b_0 is any point of B_0. Hence the function H is continuous in T_0. Hence the two functions H and h_0 are identically the same.

Expressing this result in terms of the Green's functions, we have the following theorem:

THEOREM 3. *Suppose g_k $(k = 0, 1, 2, \cdots)$ is the Green's function for the region T_k (§§ 3-4) with the pole at the same interior point P of T_0. Then the sequence $\{g_i\}$ $(i = 1, 2, \cdots)$ of functions in $T_0 - P$ converges to g_0 in $T_0 - P$, and the convergence is uniform in any closed region in $T_0 - B_0 - P$.*

6. *Critical points of Green's functions for admissible regions homeomorphic with a spherical region.* In § 3 we have defined an admissible region T_0, which has the same connectivity numbers as an ordinary spherical region but is not homeomorphic with it, and which has R_1 circles $C_1, C_2, \cdots, C_{R_1}$ in its interior linearly independent with respect to bounding in its interior. In

§ 4 we have constructed admissible regions T_i $(i = 1, 2, 3, \cdots)$ homeomorphic with an ordinary spherical region, which form a sequence converging to T_0. We shall prove the following theorem.

THEOREM 4a. *There is a positive integer K such that, for $i \geqq K$, the Green's function g_i for the region T_i with the pole at an interior point of T_0 has at least one critical point in the interior of T_0.*

If, moreover, g_i is non-degenerate in the interior of T_0, it has at least R_1 critical points of type 2 in the interior of T_0.

Proof. Let us denote by u_1 a positive lower bound of the values of g_0 on the R_1 circles $C_1, C_2, \cdots, C_{R_1}$ in the interior of T_0. From Lemma B there exist two non-critical values u' and u'' of g_0 such that

$$u_1 > u' > u'' > 0. \tag{2}$$

Let the two closed non-singular analytic regions in T_0 bounded by the surfaces $g_0 = u'$ and $g_0 = u''$ be denoted by E'_0 and E_0'' respectively (Lemma A). The circles are in the interior of E'_0, the region E'_0 is in the interior of E_0'', and the region E_0'' is in the interior of T_0 (Lemma A).

Let us confine our attention to the region E_0''. Let O denote the interior of a sufficiently small sphere about P. The closed region $E_0'' - O$ is a subregion of $T_0 - B_0 - P$. In $E_0'' - O$ the functions g_k $(k = 0, 1, 2, \cdots)$ are harmonic. Let A be any point of $E_0'' - O$. Just as we obtained (1) so we have now

$$g_i(A) - g_0(A) > 0, \qquad (i = 1, 2, 3, \cdots). \tag{3}$$

Since the sequence $\{g_i\}$ converges uniformly to g_0 in $E_0'' - O$ (Theorem 3), for a small positive constant e there exists a positive integer K such that, for $i \geqq K$,

$$g_i(A) - g_0(A) < e. \tag{4}$$

Let us assume that the positive constant e is so small that

$$u' > u'' + e.$$

From Lemma B there exists a non-critical value u_i of g_i $(i \geqq K)$ in T_i such that

$$u' > u_i > u'' + e. \tag{5}$$

Let the non-singular analytic region of points of T_i satisfying $g_i \geqq u_i$ be denoted by E_i. We shall prove the following two statements: (a) E'_0 is in the interior of E_i; and (b) E_i is in the interior of E_0''.

Let A', A_i, A'' denote any points of the surfaces $g_0 = u'$, $g_i = u_i$, $g_0 = u''$ respectively. Since A' is a point of $E_0'' - O$, from (3) we have in particular

$$g_i(A') > g_0(A') = u', \qquad (i \geqq K).$$

But from the first inequality of (5), we have

$$u' > u_i = g_i(A_i).$$

These two relations give

$$g_i(A') > g_i(A_i).$$

From Lemma A applied to g_i, this inequality shows that any point A' of $g_0 = u'$ is in the interior of E_i. The statement (a) is thus proved.

Since A'' is a point of $E_0'' - O$, from (4) we have in particular

$$g_i(A'') - g_0(A'') < e,$$

or

$$g_i(A'') < u'' + e.$$

From the second inequality of (5) we have

$$u'' + e < u_i = g_i(A_i).$$

These two relations give

$$g_i(A'') < g_i(A_i).$$

From Lemma A applied to g_i, this inequality shows that any point A'' of $g_0 = u''$ is not in the interior of E_i. The statement (b) is thus proved.

Now, since the circles $C_1, C_2, \cdots, C_{R_1}$ are in the interior of E'_0, they are in the interior of E_i from the statement (a). Since the circles are linearly independent with respect to bounding in E_0'' and since E_i is a sub-region of E_0'' from the statement (b), the circles are linearly independent with respect to bounding in E_i. Hence the first connectivity number of E_i is at least R_1. Our theorem then follows at once from Theorem 1.

COROLLARY. *There exists a closed, finite, non-singular region, homeomorphic with an ordinary spherical region, for which the Green's function with the pole at an interior point of the region has at least one critical point in the interior of the region.*

Let us note that Theorem 4a has been deduced from Theorem 1 on the bases of the Corollary to Theorem 2 and of the two properties of the sequence $\{T_i\}$ stated at the end of § 4. By the same method of proof the following general theorem can be easily established.

THEOREM 4. *Suppose D is the region in Theorem 2. Suppose $\{D_i\}$ $(i = 1, 2, 3, \cdots)$ is a sequence of admissible regions homeomorphic with an ordinary spherical region, which converges to D from the exterior and has the two properties stated at the end of § 4.*

If $R_1 \neq 0$, then Theorem 4a holds when T_i and T_0 there are replaced by D_i and D_0 respectively.

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ON OPERATIONS PERMUTABLE WITH THE LAPLACIAN.

By HILLEL PORITSKY.†

1. *Introduction.*‡ Gauss' theorem stating that the arithmetic mean $\int u dS / \int dS$ of a function u which is harmonic on and within a spherical surface S, is equal to its value at the center of S, because of its geometrical appeal and elegant simplicity, must be considered among the most attractive theorems in analysis. Koebe, Bôcher, and others have proved converse forms of Gauss' theorem by showing that this property of harmonic functions completely characterized them. Yet comparatively few results in mathematics seem to have had their starting point in that theorem, even when one considers the field of harmonic functions and functions related to them. This paper is the first one of several papers which grew out of an attempt to extend Gauss' mean value theorem.

In trying to organize the results that were first obtained it was found that they could be coördinated and the proofs simplified by the introduction of a certain linear functional operator A, presently to be defined, which is permutable with ∇^2. The proof of this permutability property, given in § 2, Theorem I, is not unlike some of the proofs that have been given for Gauss' theorem, but the more abstract result obtained can be utilized not merely to prove many laws of the spherical mean which generalize Gauss' mean value theorem, but also to derive many apparently disconnected results in the theory of harmonic and related functions, such as Bessel functions.

To describe the operator A, consider in Euclidean 3-space a concentric family of spherical surfaces imbedded in the region of definition of a given function $u(x, y, z)$, where x, y, z are rectangular Cartesian coördinates for the region. Let $\bar{u}(x, y, z)$ be a new function which is constant along each of the concentric spherical surfaces and is equal to the arithmetic mean of u over that spherical surface. The operation $A(u)$ is the operation which replaces u by $\bar{u}$:

$$A(u) = \bar{u} = \int u dS / \int dS.$$

We shall refer to this operation as "averaging" u over concentric spheres,

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‡ The results of § 2 first appeared in the author's Ph. D. thesis *Topics in Potential Theory*, Cornell 1927, and are published here for the first time.

and to A as the "averaging" operator. The permutability property of A and ∇^2 is expressed by the equation

$$\nabla^2 [A(u)] = A(\nabla^2 u).$$

This permutability is proved in Theorem I of the following section. The proof covers the n-dimensional case for which A is defined in quite a similar manner.

As the range of application of this result soon became quite extensive, it became of interest to look for other linear functional operators which are also permutable with the Laplacian. Such operators generalizing the operator A in various directions were soon found; they are dealt with in §§ 3-9; we proceed to describe them.

In §§ 3, 4 are considered operators of the form

$$L_k(u) = h_k(\omega) \int u(r, \omega') h'_k(\omega') d\omega';$$

here the integration (as in case of the operator A) is carried out over any one of the spherical surfaces of a concentric family imbedded in the domain of definition of u; r is the radius of the spheres, ω a symbolic variable for polar coördinates specifying orientation of rays through the common center, $d\omega$ the element of solid angle subtended at the center by the surface element dS: $d\omega = dS/r^2$; finally, $h_k(\omega)$, $h'_k(\omega)$ are two surface spherical harmonics of degree k, that is, functions of ω such that $h_k r^k$, $h'_k r^k$ are harmonic polynomials of degree k. It will be noticed that for $k = 0$ h_k, h'_k reduce to constants, and L_k becomes proportional to A. The nature of the operators $L_k(u)$ is rendered clear if we consider the case of two dimensions. Introducing polar coördinates, r, θ with pole at the common center and choosing $h_k = e^{ki\theta}$, $h'_k = e^{-ki\theta}/2\pi$ we find that

$$L_k(u) = e^{ki\theta} \int_0^{2\pi} u(r, \theta') e^{-ki\theta'} d\theta'/2\pi.$$

The operation L_k is thus of the nature of an operation which replaces u by one of the terms in the θ-Fourier expansion of u for each r. Likewise, for any number of dimensions the operators L_k are seen to replace u by the same type of functions as the terms in the (formal) expansion of u over each member of a family of concentric spherical surfaces in terms of a complete set of surface spherical harmonics of various degrees. Two proofs of the permutability of L_k and ∇^2 are given — one involving Euclidean operations only, the other making use of the second differential operator of Beltrami for the spherical surfaces.

The operators L_k find themselves in a sense generalized in the operators

$L_{k,m}$ of § 5. The relation of $L_{k,m}$ to L_k is sufficiently well illustrated by a particular case $(m = 1)$ if in three dimensions instead of considering a family of concentric spheres we start with a family of co-axial circles and "average" u over each circle, that is, replace u over each circle by its arithmetic mean over that circle, or, again, replace u by a term of the type occurring in the Fourier expansion of u over each circle in terms of θ, where θ is the central angle along each circle measured from a common half plane through the axis.

In § 6 we consider operators of the form

$$h(x_1, x_2, \cdots, x_m) \int_{-\infty}^{+\infty} \cdots \int h'(x'_1, x'_2, \cdots, x'_m) \times u(x'_1, x'_2, \cdots, x'_m;\ x_{m+1}, \cdots, x_n)\, dx'_1 \cdots dx'_m.$$

Here the integration is carried out over parallel m-flats immersed in a Euclidean n-space, E_n; $h(x_1, \cdots x_m)$, $h'(x_1, \cdots x_m)$ are solutions of the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2}\right)(\ ) - k(\ ) = 0,$$

where k is a constant. These operators are suggested by letting the loci of the common centers of the preceding operators $L_{k,m}$ move off to infinity. Thus for $n = 3$, $m = 1$ the integrations would extend over parallel lines or planes instead of co-axial circles or concentric spheres.

Returning to the operators

$$L_k(u) = h_k(\omega) \int h'_k(\omega') u(r, \omega')\, d\omega'$$

one might attempt to generalize them by letting $h_k(\omega)$, $h'_k(\omega)$ be *non* integer harmonics, that is, functions of ω such that for a non integer k $r^k h_k$, $r^k h'_k$ are harmonic. Such harmonics, however, prove to be no longer single valued functions of ω, but may be single valued over proper Riemann spaces spread over the unit sphere. These generalizations of L_k are considered in § 7. In § 8 we discuss the analogues of the above results for non-Euclidean spaces. Finally, several further extensions of the results of the preceding sections are considered in § 9.

As regards the proofs it may be observed that they are similar and essentially consist in an application of Green's theorem and in changing the order of differentiations and integration. A considerable part of their complexity is due to the singularity of the coördinate system at the common center of the concentric spheres.

In this paper we confine ourselves entirely to the consideration of the permutability of the operators mentioned with the Laplacian (or its proper

generalization for non-Euclidean space). The application of these results we reserve for future papers. We shall, however, illustrate the manner in which these results are applied by considering a harmonic function u; if O is any linear functional operator permutable with ∇^2, then

$$O(\nabla^2 u) = 0 = \nabla^2 [O(u)]:$$

thus $O(u)$ is also harmonic. Thus one may build new harmonic functions from a given one by applying to it any linear functional operator which is permutable with ∇^2.

2. *Permutability of the Laplacian operator* ∇^2 *with the averaging operator* A. We shall consider functions of n real variables, $x_1, x_2, \cdots, x_n$, where x_i are orthogonal coördinates of a point in n-dimensional Euclidean space E_n. The locus of points of E_n which are a constant distance r away from a fixed point we shall call a " sphere " or " spherical surface " (common terms are "$(n-1)$-sphere," " hypersphere ") and shall denote it by S. The $(n-1)$-dimensional element of " area " of S we denote by dS and its projection from the center onto a concentric unit sphere by $d\omega$; unless otherwise stated the integrals $\int f dS$, $\int f d\omega$ will extend over the whole of S. The value of $\int d\omega$, the area of a unit sphere, we denote by K_n. Finally, we write dv for the n-dimensional element of volume of E_n.

As explained in § 1 (for the case $n = 3$), by the "arithmetic mean " or " average " of $u(x_1, x_2, \cdots, x_n)$ over a spherical surface S will be understood the quotient $\int u dS / \int dS$, while the operation which consists in replacing u over each of a concentric family of spherical surfaces (lying in the domain of definition of u) by the average of u over that surface will be denoted by A. The resulting function, $A(u)$, depends on r only. In this manner $A(u)$ has been defined for $r > 0$; we extend its definition to $r = 0$ by defining $A(u)$ for $r = 0$, that is, for the center of the family, as the value of u itself at that point.

We shall now prove

THEOREM I. *The operation A of averaging over concentric spherical surfaces is permutable with the Laplacian operation ∇^2, that is,*

$$\nabla^2 [A(u)] = A [\nabla^2 u]$$

provided u is of class C'' (that is, is continuous and possesses continuous derivatives of first and second order) in the closed region bounded by concentric spheres of radii a, b, $0 < a \leqq r \leqq b$, or by the single sphere of radius b, $r \leqq b$, where r is the distance from the common center. These closed regions we denote by $R_{a,b}$, R_b, respectively.

To prove this we shall first show that the derivatives $\partial^2 A(u)/\partial x_i^2$ exist (this is implied in the statement of the theorem) and are continuous. Let $\theta_1, \theta_2, \cdots, \theta_{n-1}, r$ be a system of polar coördinates, where θ_i are constant along rays through the center; we may choose θ_i, for example, as the angles defined by

$$(1)\qquad \begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ &\cdots\cdots\cdots\cdots \\ x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \end{aligned}$$

where the origin has been put at the center. Using ω as symbolic for the $n-1$ variables θ_i, we write

$$u = u(r, \omega), \quad dS = r^{n-1}\, d\omega, \quad \int dS = r^{n-1} \int d\omega = r^{n-1} K_n.$$

We may write $A(u)$, $r = 0$ not excepted, in the form

$$A(u) = \int u(r, \omega)\, d\omega / K_n,$$

and we notice that both $d\omega$ and the limits of integration are independent of r, while $\partial u/\partial r$, $\partial^2 u/\partial r^2$ exist and are continuous in all the variables r, θ_i. Hence dA/dr, d^2A/dr^2 exist, are continuous in r in the closed interval in question, and may be obtained by differentiating under the integral sign. Moreover, the values of the former derivative for $r = 0$ is 0, since by differentiating under the integral sign we see that the directional derivatives in two opposite directions cancel each other. From this the existence and continuity of $\partial A(u)/\partial x_i$, $\partial^2 A(u)/\partial x_i^2$ except at the origin follow at once, and we may write for $r \neq 0$

$$(2)\qquad \frac{\partial A(u)}{\partial x_i} = \frac{dA(u)}{dr}\frac{x_i}{r}, \quad \frac{\partial^2 A(u)}{\partial x_i^2} = \frac{d^2A(u)}{dr^2}\frac{x_i^2}{r^2} + \frac{dA(u)}{dr}\frac{r^2 - x_i^2}{r^3}.$$

To prove the existence and continuity of the derivatives of $A(u)$ *at* the origin requires somewhat longer considerations. As regards the *existence* of the derivatives at the origin, let U be any space function depending on r only and of class C'' in r for $0 \leqq r \leqq b$. Along the x_i-axis, since $r = |x_i|$ there, we have $\partial U/\partial x_i = \pm dU/dr$ according as $x_i \gtrless 0$, while $\partial^2 U/\partial x_i^2 = d^2U/dr^2$ there for $x_i \neq 0$. Therefore at the origin $\partial U/\partial x_i$ will exist or not according as $dU/dr\,|_{r=0}$ vanishes or not. In the former case, moreover, $\partial^2 U/\partial x_i^2$ is seen to exist along the x_i-axis even for $x_i = 0$ and to be equal to $d^2U/dr^2\,|_{r=0}$ there. Applying this to $A(u)$ we see that $\partial A(u)/\partial x_i$, $\partial^2 A(u)/\partial x_i^2$ exist at the origin and

$$\left.\frac{\partial A(u)}{\partial x_i}\right|_{r=0}=0, \qquad \left.\frac{\partial^2 A(u)}{\partial x_i^2}\right|_{r=0}=\left.\frac{d^2A(u)}{dr^2}\right|_{r=0}.$$

Now as regards the *continuity* of the derivatives of $A(u)$ at the origin, if in equations (2) we let r approach zero, since $|x_i/r| \leqq 1$, we get

$$\lim_{r\to 0}\frac{\partial A(u)}{\partial x_i}=0,$$

$$\lim_{r\to 0}\frac{\partial^2 A(u)}{\partial x_i^2}=\lim_{r\to 0}\frac{d^2A(u)}{dr^2}\frac{x_i^2}{r^2}+\frac{dA(u)}{dr}\frac{r^2-x_i^2}{r^3}.$$

Putting the last term in the form

$$\frac{\dfrac{dA(u)}{dr}-\left(\left.\dfrac{dA(u)}{dr}\right|_{r=0}\right)}{r}\;\frac{r^2-x_i^2}{r^2}$$

we find

$$\lim_{r\to 0}\frac{\partial^2 A(u)}{\partial x_i^2}=\left.\frac{d^2A(u)}{dr^2}\right|_{r=0}.$$

Thus $A(u)$ is of class C'' at the origin, too.

Incidentally, it follows from the above considerations that if U is a space function which depends on r only and is of class C'' in r for $0<a\leqq r\leqq b$, then U is of class C'' in x_i in the region $R_{a,b}$, while if U is of class C'' in r for $0\leqq r\leqq b$, then U need not be even of class C' in R_b, unless $dU/dr\,|_{r=0}=0$, in which case U will be of class C'' in R_b. The Laplacian of such a function will be given by

$$\nabla^2U=\frac{d^2U}{dr^2}+\frac{n-1}{r}\frac{dU}{dr}=r^{n-1}\frac{d}{dr}\left(r^{n-1}\frac{dU}{dr}\right) \quad \text{for} \quad r>0, \tag{3}$$

$$\nabla^2U=n\,\frac{d^2U}{dr^2} \quad \text{for} \quad r=0. \tag{3'}$$

We now proceed with the proof by applying Gauss' Theorem to the spherical shell bounded by two spheres S_1, S_2 of radii r_1, r_2, $0<a\leqq r_1<r_2\leqq b$:

$$\int \nabla^2u\,dv=\int\frac{\partial u(r_2,\omega)}{\partial r}\,dS_2-\int\frac{\partial u(r_1,\omega)}{\partial r}\,dS_1$$

$$=r_2^{\,n-1}\int\frac{\partial u(r_2,\omega)}{\partial r}\,d\omega-r_1^{\,n-1}\int\frac{\partial u(r_1,\omega)}{\partial r}\,d\omega.$$

Dividing by r_2-r_1, letting one of the radii approach the other, replacing the latter by r, and denoting the corresponding spherical surface by S we get

$$(4) \qquad \int \nabla^2 u(r,\omega)\,dS = \frac{d}{dr}\left(r^{n-1}\int \frac{\partial u(r,\omega)}{\partial r}\,d\omega\right).$$

From this follows upon dividing by $\int dS = K_n r^{n-1}$

$$A(\nabla^2 u) = r^{1-n}\,\frac{d}{dr}\left[r^{n-1}\int \frac{\partial u(r,\omega)}{\partial r}\,\frac{d\omega}{K_n}\right].$$

But if we interchange the order of integration and one differentiation and take account of (3), we find that the right hand member above reduces to $\nabla^2 A(u)$. The theorem is thus proved except for $r=0$. In this case it follows from the continuity of both functions $A[\nabla^2 u]$, $\nabla^2[A(u)]$.

3. *Permutability of the operators L_k with the Laplacian.* We shall now extend the results of § 2 by showing that the property there proved for the averaging operator (namely, its permutability with the Laplacian) is also possessed by the operators L_k defined as follows:

Let $H_k(x_1, x_2, \cdots, x_n)$, $H_k'(x_1, x_2, \cdots, x_n)$ be two homogeneous harmonic polynomials of degree k, or in familiar teminology, two (solid) spherical harmonics of degree k, and let $h_k = H_k/r^k$, $h_k' = H_k'/r^k$ be the corresponding "surface" spherical harmonics. The latter are independent of r, being, in fact, polynomials in $\cos\theta_1, \sin\theta_1, \cdots, \sin\theta_{n-1}$. Using the notation of the preceding sections we shall write $h_k = h_k(\omega)$, $h_k' = h_k'(\omega)$. Corresponding to two such spherical harmonics of degree k we define $L_k(u)$:

$$(5) \qquad L_k(u) = h_k(\omega)\int h'_k(\omega')u(r,\omega')\,d\omega'.$$

The result of applying L_k to a function u is thus a new function which is a product of the surface spherical harmonic h_k by a function of r; for brevity we shall denote the latter by $I(r)$:

$$I(r) = \int h_k(\omega')u(r,\omega')\,d\omega'.$$

At the origin we define $L_k(u)$ for $k \neq 0$ as equal to zero. For $k=0$ h_k, h_k' are to be replaced by constants and L_0 is seen to be proportional to $A(u)$; this we assume to hold at the origin, too (that is, with the *same* constant of proportionality).

As pointed out in § 1, the functions $L_k(u)$ are of the same type as the functions occurring in the formal expansion of u along each member of the spheres $r=$ const. in terms of a complete set of surface spherical harmonics independent of r. In this connection the function $I(r)$ appears to be a "Fourier constant" for each spherical surface and the function $L_0(u)$ corresponds to the constant term of the above expansion.

For these L_k we now state

THEOREM II. *The operators L_k are permutable with ∇^2, that is,*

$$\nabla^2 [L_k(u)] = L_k [\nabla^2 u] \tag{6}$$

under the same conditions on u as in Theorem I.

Since Theorem I is contained in Theorem II as a special case and the proof which follows could be rendered independent of the preceding section, the latter could have been omitted. A separate proof of Theorem I seemed, however, desirable in view of its simplicity and its importance for applications, as well as in order to break up the not inconsiderable complexity of the subject matter.

We begin with the consideration of $\partial I/\partial x_i$, $\partial^2 I/\partial x_i^2$. Their existence and continuity for $r > 0$ follows from considerations similar to those of the preceding section. Hence $L_k(u)$ is of class C'' except possibly at the origin.

We proceed to compute $\nabla^2 L_k(u)$:

$$\begin{aligned}\nabla^2 [L_k(u)] &= \nabla^2(h_k I)\\ &= \nabla^2(H_k I r^{-k})\\ &= \nabla^2 H_k(Ir^{-k}) + 2\nabla H_k \cdot \nabla(Ir^{-k}) + H_k \nabla^2(Ir^{-k}),\end{aligned}$$

where the second term is twice the scalar product of the gradients ∇H_k, $\nabla(Ir^{-k})$; this term may be replaced by $2\{\partial H_k/\partial r)\ [d(Ir^{-k})/dr]\}$ since the gradient of Ir^{-k} points in a radial direction. On equating $\nabla^2 H$ to zero, replacing H_k by $r^k h_k$ and ∇^2 in the last term by (3), and carrying out the differentiations we get

$$\nabla^2 [L_k(u)] = h_k \left[\frac{d^2 I}{dr^2} + \frac{n-1}{r}\frac{dI}{dr} + \frac{k(2-n-k)}{r^2}\, I\right]. \tag{7}$$

Next consider

$$L_k(\nabla^2 u) = h_k(\omega) \int h'_k(\omega')\nabla^2 u(r, \omega')\, d\omega'.$$

By applying Green's Theorem to the volume between two concentric spherical surfaces which are allowed to approach each other we obtain a result generalizing equation (4):

$$\int (v\nabla^2 u - u\nabla^2 v)\, dS = (d/dr)\,[r^{n-1} \int (v\, \partial u/\partial r - u\, \partial v/dr)\, d\omega]; \tag{8}$$

here v as well as u is of class C''. If in this equation we put H_k' in place of v, we get

$$\int H'_k \nabla^2 u\, dS = (d/dr)\,[r^{n-1} \int (H'_k \partial u/\partial r - u\partial H'_k/\partial r)\, d\omega],$$

and, replacing H'_k by $h'_k r^k$ and dS by $r^{n-1}\,d\omega$,

$$r^{k+n-1}\int h'_k\nabla^2 u d\omega = (d/dr)\{r^{n+k-1}[\int h'_k(\partial u/\partial r)d\omega - (k/r)\int h'_k u d\omega]\};$$

finally, changing the order of differentiation and integration in $\int h'_k(\partial u/\partial r)d\omega$, solving for $\int h'_k\nabla^2 u d\omega$, and simplifying, we obtain for this integral the bracket of the right-hand member of (7). The proof of Theorem II, except for $r=0$, is thus complete.

For $r=0$ we may verify the theorem directly for the cases where u is a polynomial of the second degree in $x_1,\cdots x_n$; it remains to prove it for functions u of class C'' and such that for small r

$$u = o(r^2),\quad \partial u/\partial x_i = o(r),\quad \partial^2 u/\partial x_i^2 = o(1).$$

First we point out that since $h_k = O(1)$,

$$I = \int h'_k(\omega')u(r,\omega')d\omega' = o(r^2),$$

$$dI/dr = \int h'_k(\omega')[\partial u(r,\omega')/\partial r]\,d\omega' = o(r),\quad d^2I/dr^2 = o(1);$$

hence

$$\frac{\partial I}{\partial x_i} = \frac{dI}{dr}\frac{x_i}{r} = o(r),\ \frac{\partial^2 I}{\partial x_i^2} = \frac{dI}{dr}\frac{r^2 - x_i^2}{r^3} + \frac{d^2I}{dr^2}\frac{x_i}{r} = o(1).$$

Again, $\partial h_k(\omega)/\partial x_i = O(r^{-1})$, $\partial^2 h_k(\omega)/\partial x_i^2 = O(r^{-2})$, hence $L_k(u) = h_k I = o(r^2)$ and $L_k(u)$ is continuous at the origin; its partial derivatives with respect to x_i vanish there, for

$$\partial L_k(u)/\partial x_i\,|_{r=0} = \lim_{r\to 0} O(r^2)/x_i = \lim_{r\to 0} O(r) = 0.$$

Near the origin

$$\partial L_k(u)/\partial x_i = (\partial h_k/\partial x_i)I + h_k\partial I/\partial x_i = O(r^{-1})o(r^2) + O(1)o(r) = o(r);$$

$\partial L_k(u)/\partial x_i$ are thus continuous at $r=0$. Likewise by using the above order relations for the derivatives of I and of h_k, one proves that $\partial^2 L_k(u)/\partial x^2_i$ are continuous at $r=0$ and vanish there. The last statement also applies, however, to $L_k(\nabla^2 u)$ since

$$L_k(\nabla^2 u) = h_k\int h'_k\nabla^2 u d\omega = o(1).$$

We have thus shown that both members of (6) are continuous at the origin and are equal to each other there, too. The proof of Theorem II is thus complete.

The cross derivatives $\partial^2 L_k(u)/\partial x_i\,\partial x_j$ may be shown to be continuous inclu-

sive of the origin in the same manner as was done for $\partial^2 L_k(u)/\partial x_i^2$. Hence $L_k(u)$ is of class C''.

4. *Another proof of Theorem II; the second differential operator of Beltrami.* The proof of Theorem II admits (for $r > 0$) of another form which is illuminating, and reveals the *raison d'être* of the theorem.

Recall expression (3) for ∇^2. Replacing the r- differentiations in it by partial differentiations, we shall denote the result by D_2:

$$(9) \qquad D_2(u) = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} = r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right),$$

and write

$$(10) \qquad \nabla^2 u = D_2 u + \Delta_2 u.$$

The operator Δ_2 is obviously independent of axes and coördinates; the following properties of this operator will be utilized:

1. Δ_2 is a sum of homogeneous differential operators of first and second orders; if polar coördinates r, θ_i are used, Δ_2 involves θ_i—differentiations only.

2. For any two functions $u(\omega)$, $v(\omega)$ which are single valued over a sphere and of class C'' the equation holds

$$(11) \qquad \int [v\Delta_2 u - u\Delta_2 v]\, d\omega = 0.$$

3. A surface spherical harmonic of degree k, h_k, satisfies the equation

$$(12) \qquad \Delta_2(h_k) = -k(k+n-2)h_k r^{-2}.$$

Granting these properties, we shall now prove that ∇^2 and L_k are permutable by showing that L_k is permutable with each of the operators D_2, Δ_2 whose sum is equal to ∇^2. Indeed,

$$D_2[L_k(u)] - L_k(D_2 u) = r^{1-n}(\partial/\partial r\{r^{n-1}(\partial/\partial r)[h_k(\omega)\int h'_k(\omega)u(r,\omega')d\omega']\}$$
$$- h_k(\omega)\int h'_k(\omega')r^{1-n}(\partial/\partial r\{r^{n-1}[\partial u(r,\omega')/\partial r]\}d\omega'$$

and this vanishes since the r-differentiations and the ω-integrations are permutable. Again,

$$\Delta_2[L_k(u)] - L_k(\Delta_2 u) = \Delta_2[h_k(\omega)\int h'_k(\omega')u(r,\omega')d\omega']$$
$$- h_k(\omega)\int h'_k(\omega')\Delta_2[u(r,\omega')]\, d\omega'.$$

The right hand member may now be transformed by noticing that in the first term the integral is a function of r only (denoted by $I(r)$ in the preceding section), while Δ_2 involves differentiations along the surface of the sphere only, and utilizing (11); we get

$$\Delta_2\,[L_k(u)] - L_k(\Delta_2 u) = [\Delta_2 h_k(\omega)] \int h'_k(\omega')u(r,\omega')\,d\omega' - h_k(\omega)\int u(r,\omega')\Delta_2\,[h'_k(\omega')]\,d\omega'.$$

It only remains to make use of (12) to reduce $\Delta_2[L_k(u)] - L_k(\Delta_2 u)$ to zero.

We now turn to the proof of the properties of Δ_2 which have been utilized above. The first one may be proved in a straightforward manner by expressing ∇^2 in terms of r- and θ-differentiations by means of the familiar expression of ∇^2 in curvilinear coördinates. Equation (12) now follows from

$$0 = \nabla^2(H_k) = \nabla^2(h_k r^k)$$

by replacing ∇^2 by the right hand member of (10), noting that D_2 operates only on r^k and Δ_2 only on h_k, and carrying out the r-differentiations involved. Finally, as regards equation (11), it is essentially equivalent to equation (8) of the preceding section. This may be seen by replacing dS in the left hand member of that equation by $r^{n-1}d\omega$, dividing both sides by r^{n-1}, and interchanging the order of the integration and the subsequent operations on the right; we get

$$\begin{aligned}\int (v\nabla^2 u - u\nabla^2 v)\,d\omega &= \int r^{1-n}(\partial/\partial r)\,[r^{n-1}(v\partial u/\partial r - u\partial v/\partial r)]\,d\omega\\ &= \int [(n-1)/r](v\partial u/\partial r - u\partial v/\partial r)\,d\omega + \int\left(\frac{\partial v}{\partial r}\frac{\partial u}{\partial r} - \frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\right)d\omega\\ &\quad + \int (v\partial^2 u/\partial r^2 - u\partial^2 v/\partial r^2)\,d\omega\\ &= \int (vD_2 u - uD_2 v)\,d\omega.\end{aligned}$$

Transposing and replacing $\nabla^2 - D_2$ by Δ_2 we obtain the formula in question.

Another method of deducing the properties of the operator Δ_2 is also of interest. This operator constitutes for the sphere what is known as the "second differential invariant operator", due to Beltrami. For any Riemannian space with a metric

$$ds^2 = \sum_{i,j} g_{ik}\,d\xi_i\,d\xi_j \qquad (i, j = i, \cdots, n),$$

the latter operator might be defined by

$$(13)\quad \Delta_2(u) = g^{-1/2} \sum_i (\partial/\partial\xi_i)[g^{1/2} \sum_j g^{ij}(\partial u/\partial\xi_j)] \qquad (i, j = 1, \cdots, n);$$

here g is the absolute value of the determinant $|g_{ij}|$ ($g\, d\xi_i \cdots d\xi_n$ is thus the element of volume $d\tau$ of R_n), while g^{ij} is the matrix which is "reciprocal" to the matrix g_{ij}. In particular, if ξ_i are orthogonal coördinates, so that

$$ds^2 = \sum h_i^2\, d\xi_i^2 \qquad (i = 1, \cdots, n);$$

(13) becomes

$$(13')\quad \Delta_2 = \frac{1}{h_i \cdots h_n} \sum_i \frac{\partial}{\partial\xi_i}\left(\frac{h_1 \cdots h_{i-1}\, h_{i+1} \cdots h_n}{h_i} \frac{\partial u}{\partial\xi_i}\right).$$

This operator may be proved to possess a "Green" theorem similar to that possessed by the Laplacian ∇^2 (to which it reduces if R_n is reduced to E_n):

$$(14)\quad \int (v\Delta_2 u - u\Delta_2 v)\, d\tau = \int [v(\partial u/\partial n) - u(\partial v/\partial n)]\, d\Sigma,$$

where the left hand integral extends over a region τ of R_n and the right hand integral over its boundary Σ.

Now equation (11) constitutes a special instance of this general integration theorem: if for R_n we choose a sphere S in E_n and apply (14) to the complete sphere, the right hand member reduces to zero since there is no boundary, and we get

$$\int (v\Delta_2 u - u\Delta_2 v)\, dS = 0;$$

from this (11) follows by dividing by r^{n-1} provided that the identity of the present operator Δ_2 with the operator Δ_2 as defined by (10) is granted.

The identity of these two operators may be established without resorting to detailed computations. If in (14) we put $v = 1$, it reduces to

$$(14')\quad \int \Delta_2 u\, d\tau = \int (\partial u/\partial n)\, d\Sigma.\dagger$$

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† From this one deduces the well known definition of Δ_2 as the limit of the ratio $\int (\partial u/\partial n)\, d\tau/\tau$ as τ shrinks to a point; the invariant nature of Δ_2 as regards coördinates is thus manifest.

It will be recalled that the first invariant differential parameter Δ_1, due to Lamé, is the maximum of the various directional derivatives of u at a point. A further intrinsic definition of Δ_2 is through the variation of the integral $\int \Delta_1 u\, dt$, thus:

$$\delta \int \Delta_1(u)\, d\tau = \int \Delta_2 u\, \delta u\, d\tau$$

for variations δu of u, which vanish on the boundary of τ. See, for instance, Courant Hilbert, *Methoden der Mathematischen Physik*, Vol. 1, p. 194; also W. Blaschke, *Vorlesungen über Differentialgeometrie*, Vol. 1, §§ 66-68.

Next apply Gauss' theorem $\int \nabla^2 u \, dv = \int (\partial u/\partial n) dS$ to the volume bounded by two concentric spherical surfaces $r = r_1$, $r = r_2$, and a solid cone obtained by letting ω range over a proper region ω_R of the unit sphere. If we divide by $(r_2 - r_1)$ and let r_1, r_2 approach r, the left-hand member approaches $\int_{S_R} \nabla^2 u dS$, where S_R is the portion of S for which ω lies in ω_R; the contribution to the surface integral from the two concentric surfaces, upon dividing by $r_2 - r_1$, is seen to approach

$$\frac{d}{dr}\left(r^{n-1}\int_{\omega_R}\frac{\partial u(r,\omega)}{\partial r}\,d\omega\right) = \int_{\omega_R}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial u(r,\omega)}{\partial r}\right]d\omega$$

$$= \int_{S_R} r^{1-n}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial u(r,\omega)}{\partial r}\right]dS = \int_{S_R}\left(\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r}\right)dS;$$

finally, the contribution to the surface integral from the boundary of the solid angle gives rise to a similar $[(n-2)$-dimensional$]$ integral over the boundary of S_R, involving derivatives of u in directions normal to this boundary and tangent to S; by means of (14′) this integral may be changed to $\int_{S_R} \Delta_2 u \, dS$. Hence we get

$$\int_{S_R} \nabla^2 u \, dS = \int_{S_R} \{\partial^2 u/\partial r^2 + [(n-1)/r](\partial u/\partial r)\} dS + \int_{S_R} \Delta_2 u \, dS,$$

and on equating the integrands obtain equation (10). We have thus established the identity of the two ways of introducing Δ_2.

5. *Extension of the property of permutability with the Laplacian to the operators* $L_{k,m}$. We now consider a Euclidean space E_{n+m} of $n+m$ dimensions with the rectangular coördinates $x_1, \cdots, x_n;\ x_{n+1}, \cdots, x_{n+m}$, and a function $u(x_1, \cdots, x_n;\ x_{n+1}, \cdots, x_{n+m})$ of class C''. If we equate $x_{n+1}, \cdots, x_{n+m}$ to the constants $C_{n+1}, \cdots, C_{n+m}$, we find ourselves in a Euclidean space E_n of n dimensions. To the resulting function $u(x_1, \cdots, x_n;\ C_{n+1}, \cdots, C_{n+m})$ we may apply the operators L_k of the last section, obtaining thereby a new function over E_n. If we now vary the constants $C_{n+1}, \cdots, C_{n+m}$ while we keep fixed the harmonics H_k, H'_k in terms of which L_k is defined † and transform u into $L_k(u)$ over each of the E_n thus obtained, there results a new function over E_{n+m}; this function we denote by $L_{k,n}(u)$. We shall now show that these operators $L_{k,n}$ are permutable with

† These harmonics involve $x_1, \cdots, x_n$ only.

$$\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x^2_{n+m}.$$

Indeed, by Theorem II $L_{k,n}$ is permutable with $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$;

it remains to show that it is also permutable with $\partial^2/\partial x^2_{n+1} + \cdots + \partial^2/\partial x^2_{n+m}$. Now for $r > 0$ L_k is given by

$$(5) \qquad L_k(u) = h_k(\omega) \int h'_k(\omega')u(r, \omega'; x_{n+1}, \cdots, x_{n+m})\, d\omega'$$

where $r^2 = x_1{}^2 + \cdots + x_n{}^2$ and ω stands for $\theta_1, \cdots, \theta_{n-1}$ defined—say—by equations (1). Thus both $h_k(\omega)$, $h'_k(\omega')$ as well as the limits of integration are independent of $x_{n+1}, \cdots, x_{n+m}$; because of this and since $\partial u/\partial x_{n+1}$ is continuous in all the variables $r, \theta_1, \cdots, \theta_{n+1}$; $x_{n+1}, \cdots, x_{n+m}$, it follows that $\partial L_{k,n}/\partial x_{n+1}$ exists and may be obtained by differentiating under the integral sign; there, of course, the differentiation is applied to u only; we have thus shown that for $r > 0$ $L_{k,n}$ is permutable with $\partial/\partial x_{n+1}$; likewise it is proved to be permutable with $\partial/\partial x_{n+2}, \cdots, \partial^2/\partial x^2_{n+1}$, $\partial^2/\partial x^2_{n+2}, \cdots$, and hence with $\partial^2/\partial x^2_{n+1} + \cdots + \partial^2/\partial x^2_{n+m}$.

Now consider the locus $r = 0$. Denote the operator $\partial^2/\partial x^2_{n+1} + \cdots + \partial^2/\partial x^2_{n+m}$ by $\bar{\nabla}^2$. The function $\bar{\nabla}^2 u$ is continuous inclusive of $r = 0$. Hence we infer from Theorem II that $L_k(\bar{\nabla}^2 u)$ is continuous in $x_1, \cdots, x_n$ at each point of $r = 0$. Along the latter locus, however, $L_k(\bar{\nabla}^2 u)$ reduces by definition either to zero ($k > 0$) or to a constant multiple of $\bar{\nabla}^2 u$ ($k = 0$), and is thus continuous *within* $r = 0$. $L_k(\bar{\nabla}^2 u)$ is therefore continuous at $r = 0$. Likewise $L_k(u)$ reduces for $r = 0$ either to zero or to the same constant multiple of u. Therefore $\bar{\nabla}^2 L_k(u)$, already proved existent and equal to $L_k(\bar{\nabla}^2 u)$ for $r > 0$, is seen to exist at $r = 0$ and is equal to $L_k(\bar{\nabla}^2 u)$ there, too.

In this way is proved the permutability of $L_{k,n}$ with ∇^2 for regions given by

$$a(x_{n+1}, \cdots, x_{n+m}) \leqq r \leqq b(x_{n+1}, \cdots, x_{n+m}),$$

where a, b are continuous functions of $x_{n+1}, \cdots, x_{n+m}$. For more general regions of E_{n+m}, obtained by letting $r, x_{n+1}, \cdots, x_{n+m}$ range over an $(m + 1)$-dimensional region, the proof may be carried out by approximating to the boundary by means of a finite number of regions whose boundaries are of the above type. For brevity we shall refer to such regions (and to regions obtained from them by a rigid movement of space) as "*regions over which $L_{k,n}$ is applicable.*"

If we now revert to employing n for the *total* number of dimensions of the Euclidean space under consideration and replace the above n by m ($\leqq n$), we may summarize the above result in a theorem which forms an extension of Theorem II:

THEOREM III. *The operators $L_{k,m}$ are permutable with the Laplacian, that is,*

$$\nabla^2[L_{k,m}(u)] = L_{k,m}(\nabla^2 u)$$

provided u is of class C'' in a region over which $L_{k,m}$ is applicable.

6. *Permutability of the Laplacian with the operators $L^*_{k,m}$.* We shall now consider the operators $L^*_{k,m}$ defined by

$$(15)\quad L^*_{k,m}(u) = h_k(x_1, x_2, \cdots, x_m) \int \overset{+\infty}{\underset{-\infty}{\cdots}} \int h'_n(x'_1, x'_2, \cdots, x'_m)$$
$$\times\, u(x'_1, x'_2, \cdots, x'_m;\, x_{m+1}, \cdots, x_n)\, dx'_1, \cdots, dx'_m,$$

where $h_k(x_1, \cdots, x_m)$, $h_k'(x_1, \cdots, x_m)$ are solutions of

$$(\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_m^2) h = kh$$

for a constant k for $-\infty < x_1, \cdots, x_m < +\infty$.

It is of interest to point out (though unnecessary for the proof which follows) that these operators may be considered as the limits approached by the operators L_k, $L_{k,m}$ of the preceding sections when the centers of the spheres or subspheres recede to infinity and the degree of the surface harmonics h_k, h'_k is properly increased. Thus, if for definiteness, we consider in E_3 a concentric family of spheres with an associated operator

$$(5)\qquad L_k(u) = h_k(\omega) \int h'_k(\omega') u(r, \omega') \, d\omega'$$

and let the center recede to infinity, say, in the direction of the x_3-axis, the spheres $r =$ constant flatten out into the planes $x_3 =$ constant. Now it was shown in § 4 that along a sphere of radius r surface harmonics $h_k(\omega)$ of degree k satisfy the equation,

$$(12)\qquad \Delta_2\, h_k = - k(k + n - 2) r^{-2} h_k$$

(with $n = 3$). Therefore, if at the same time that the center recedes to infinity we let k become infinite so that $-k(k + n - 2)r^{-2}$ approaches a finite limit k', we are led to consider the equation

$$(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2) h + k'h = 0$$

as the limit of (12), and the operator L^*_k as the limit of L_k.

A familiar instance of the above limiting process is the passage from the Legendre polynomials to Bessel's function:

$$\lim_{n\to\infty} P_n(\cos\theta/n) = J_0(\theta).\dagger$$

† See, for instance, G. N. Watson, *Bessel Functions*, p. 155.

Returning to the general operator $L^*_{k,m}$ defined by (15), we now state:

THEOREM IV. *The operators $L^*_{k,m}$ are permutable with ∇^2 provided that*

1. *u is of class C'' in any region R which is the product complex of any finite region R_m in the $(x_1, x_2, \cdots, x_m)$—space by a finite region R_{n-m} in the $(x_{m+1}, \cdots, x_n)$—space;*

2. *The integrals*

$$I_1 = \int \overset{+\infty}{\underset{-\infty}{\cdots}} \int h'_k(x_1, \cdots, x_m) u(x_1, \cdots, x_n)\, dx_1, \cdots, dx_m,$$

$$I_2 = \int \overset{+\infty}{\underset{-\infty}{\cdots}} \int h'_k(x_1, \cdots, x_m)\,(\partial^2/\partial x^2_{m+1} + \cdots + \partial^2/\partial x_n^2) u(x_1, \cdots, x_n)\, dx_1 \cdots dx_m,$$

$$I_3 = \int \overset{+\infty}{\underset{\infty-}{\cdots}} \int h'_k(x_1, \cdots, x_m)\,(\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_m^2) u(x_1, \cdots, x_n)\, dx_1 \cdots dx_m$$

converge.

3. *The order of the differentiations and integrations occurring in the integral I_2 may be inverted, that is,*

$$(16) \quad (\partial^2/\partial x^2_{m+1} + \cdots \partial^2/\partial x_n^2) \int \overset{-\infty}{\underset{+\infty}{\cdots}} \int h'_k(x_1, \cdots, x_m) u(x_1, \cdots, x_n)\, dx_1 \cdots dx_m$$

$$= \int \overset{+\infty}{\underset{-\infty}{\cdots}} \int h'_k(x_1, \cdots, x_m)\,(\partial^2/\partial x^2_{m+1} + \cdots + \partial^2/\partial x_n^2) u(x_1, \cdots x_n)\, dx_1 \cdots dx_m.\dagger$$

4. *The integral*

$$\int_{S_{m-1}} (u\, \partial h'_k/\partial n - h'_k\, \partial u/\partial n)\, dS_{m-1}$$

extended over the boundary S_{m-1}, of the region R_m, where $\partial/\partial n$ denotes the derivative in the direction of the outer normal, approaches zero as R_m expands to infinity, that is, as it expands so as to enclose any point of the $x_1, \cdots, x_m$-space.

To prove this break up the operator ∇^2 into two parts:

$$\nabla^2 = (\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_m^2) + (\partial^2/\partial x^2_{m+1} + \partial^2/\partial x_n^2) = {}'\nabla^2 + {}''\nabla^2.$$

We shall show that each part is permutable with $L^*_{k,m}$.

First consider the operator $''\nabla^2$. Conditions for its permutability with

† A sufficient condition for this is that the right hand member of (16) converge uniformly with respect to $x_{m+1}, \cdots, x_n$. See de la Vallée Poussin, *Cours d'Analyse*, Vol. 2, § 23.

$L^*_{k,m}$ are essentially contained in condition 3 above. Thus, equation (16) implies that $''\nabla^2$ may be applied to the integral

$$I_2 = \int_{-\infty}^{+\infty}\cdots\int h'_k(x'_1, \cdots, x'_m)u(x'_1, \cdots, x'_m;\ x_{m+1}, \cdots, x_n)\,dx'_1 \cdots dx'_m;$$

hence it may also be applied to $h_k(x_1, \cdots, x_m)\, I_2$ yielding

$$''\nabla^2(h_k I_2) = h_k\, ''\nabla^2 I_2.$$

Multiplying both members of (16) by h_k and transforming the resulting left hand member by means of the last equation, we obtain

$$''\nabla^2 L^*_{k,m}(u) = L^*_{k,m}(''\nabla^2 u).$$

Next consider the operator $'\nabla^2$.

$$\begin{aligned} L^*_{k,m}('\nabla^2 u) &= h_k(x_1, \cdots, x_m)\int_{+\infty}^{-\infty}\cdots\int h'_k(x'_1, \cdots, x'_m) \\ &\quad\times (\partial^2/\partial x_1'^2 + \cdots + \partial^2/\partial x_m'^2)u(x'_1, \cdots, x'_m;\ x_{m+1}, \cdots, x_n)\,dx'_1 \cdots dx'_m \\ &= h_k \lim_{R_m\to\infty}\int_{R_m} h'_k\, '\nabla^2 u\, dv_m, \end{aligned}$$

where the notation is obvious; the integral involved is precisely I_3; it may be transformed by means of Green's theorem thus:

$$\int_{R_m} h'_k\, '\nabla^2 u\, dv_m = \int_{R_m} ('\nabla^2 h'_k)u\, dv_m + \int_{S_{m-1}} (h'_k\, \partial u/\partial n - u\, \partial h'_k/\partial n)\, dS_{m-1}.$$

Now on account of condition 4, the last integral approaches zero as R_m expands to infinity. Hence,

$$\begin{aligned} L^*_{k,m}('\nabla^2 u) &= h_k \lim_{R_m\to\infty}\int_{R_m} ('\nabla^2 h'_k)u\, dv_m \\ &= kh_k \lim_{R_m\to\infty}\int_{R_m} h'_k\, u\, dv_m \\ &= kh_k I_1. \end{aligned}$$

On the other hand,

$$'\nabla^2 L^*_{k,m}(u) = '\nabla^2(h'_k I_1) = ('\nabla^2 h'_k)I = kh'_k I_1$$

since I_1 is independent of $x_1, \cdots, x_m$. Hence $L^*_{k,m}$ is permutable with the operator $'\nabla^2$ as well. The proof of Theorem IV is thus complete.

7. *Generalization to multiple-valued harmonics.* Returning, say, to the operators L_k of §§ 3, 4, one may note on examining the proof of Theorem II

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that while use is made of the fact that $h_k(\omega)r^k$, $h'_k(\omega)r^k$ are harmonic, no explicit use is made of the facts that k is an integer, and that these functions are polynomials in $x_1, \cdots, x_n$. The proof, it appears, would go thru just as well if for a *non*-integer constant k the functions $h_k(\omega)r^k$, $h'_k(\omega)r^k$ were harmonic, or, what amounts to the same thing, if for such a constant k the functions $h_k(\omega)$, $h_k'(\omega)$ satisfied the equation (12)

$$\Delta_2 h_k = -k(k+n-2)h_k$$

along the unit sphere. However, such an extension of Theorem II, while possible, is really vacuous due to the circumstance that the only values of the parameter κ for which there exists a non-zero solution of

$$\Delta_2 h = \kappa h$$

along a unit sphere (in E_n) are precisely the values $\kappa = -k(k+n-2)$, where κ is an integer, and that for any such characteristic parameter value the function hr^k is a harmonic polynomial of degree k. For this reason the direct extension of Theorem II to non-integer k is impossible.

To get an idea of how the above results may be extended to more general surface harmonics, that is, to general solutions of (12) for which $r^k h_k$ are not polynomials, consider the case $n = 2$. In this case equation (12) reduces along the unit circle to $d^2h/d\theta^2 = -k^2h$; the solutions of the latter equations are linear combinations of $e^{ki\theta}$, $e^{-ki\theta}$ and are thus single-valued over the unit circle if and only if k is an integer. For other values of k these solutions may be considered to be single-valued over a proper closed or open curve Ω which covers the unit circle a finite or infinite number of times. The operator

$$h_k(\theta)\int h'_k(\theta')u(r,\theta')\,d\theta'$$

may now be applied to any function u which is similarly multiple valued for each r; that is, single-valued over the Riemann surface which is the product of Ω by an r-interval. The proof of Theorem II may be easily adapted to the case of rational k.

It will be recalled that in the proof of § 4, ∇^2 is broken up into D_2 and Δ_2, and each of these is shown to be permutable with L_k. The permutability of D_2 follows from a change of the order of its r-differentiations and ω-integrations; that of Δ_2 from equation (11), itself a special case of the Green-Beltrami Theorem (14). For $n = 2$, we have

$$D_2 = \partial^2 u/\partial r^2 + (1/r)(\partial u/\partial r), \qquad \Delta_2 = \partial^2 u/r^2\partial\theta^2,$$

and (11), (14) reduce to

$$\int_0^{2\pi}(v\,\partial^2u/r^2\partial\theta^2 - u\,\partial^2v/r^2\partial\theta^2)\,d\theta = 0,$$

$$\int_{\theta_1}^{\theta_2}(v\,\partial^2u/r^2\partial\theta^2 - u\,\partial^2v/r^2\partial\theta^2)\,d\theta = v\,\partial u/r\partial\theta - u\,\partial v/r\partial\theta\Big]_{\theta_1}^{\theta_2}$$

respectively.

It will be seen that for rational k, $k = h/l$, where h and l are relatively prime integers, the proof applies provided that the limits 0 and 2π are placed by 0 and $2\pi l$. However, for irrational k, the operator in question becomes

$$h_k(\theta)\int_{-\infty}^{\infty}h'_k(\theta')u(r,\theta')\,d\theta',$$

and proper additional conditions have to be introduced due to the infinite limits; this may be done in a fashion quite analogous to that of the preceding section in connection with $L_{k,m}$ for which the range of integration was infinite.

In quite a similar manner one could treat for any n the operator

$$\Lambda_k(u) = h_k(\Omega)\int h'_k(\Omega')u(r,\Omega')\,d\Omega',$$

where h_k, h_k' are *any* two solutions of the equation (12)

$$\Delta_2 h = kh$$

over the unit sphere, and Ω, Ω' are the Riemann surfaces spread over the unit sphere over which h_k, h_k' are respectively single-valued; the function u is single-valued over the product complex of Ω' by an r-interval, $0 < a \leqq r \leqq b$. For $n = 2$, Ω, Ω' necessarily possess the same structure; such need not be the case, however, for $n > 2$. The Riemann surfaces Ω, Ω' are not to be confused with the spaces of Riemann differential geometry but are to be thought of as consisting of a finite or infinite number of possibly incomplete copies of the unit sphere, properly cut and cross-connected. Points of Ω' at which h'_k is non-analytic along the unite sphere will be considered as belonging to the boundary of Ω'; this boundary will include points in whose neighborhood h'_k fails to be single-valued along the unit sphere. Points not on the boundary of Ω' will be called "inner" points.

The conditions for the permutability of Λ_k and ∇^2, formulated below in Theorem V, are quite analogous to those of Theorem IV of the preceding section. The analogue of the finite region R_m is taken by a part Ω'_f of Ω', which consists of a finite number of possibly incomplete copies of the unit sphere, and contains no boundary point of Ω' either within it or on its boundary; we shall call Ω'_f a "finite part" of Ω'.

To avoid unnecessary repetitions Theorem V is stated for the operators $\Lambda_{k,m}$ defined by

$$\Lambda_{k,m}(u) = h_k(\Omega) \int h'_k(\Omega') u(r, \Omega'; x_{m+1}, \cdots, x_n) d\Omega';$$

and which reduce to the operators just described, but for an m-dimensional case, by putting $x_{m+1}, \cdots, x_n$ equal to constants. Ω, Ω' are Riemann surfaces spread over the "sub-spheres" $x_1^2 + x_2^2 + \cdots + x_m^2 = r^2$. These operators extend the operators $L_{k,m}$ of § 5 in the same way that Λ_k extend the operators L_k, and reduce to Λ_k for $m = n$.

THEOREM V. *The operators $\Lambda_{k,m}$ are permutable with ∇^2 except where $h_k(\Omega)$ is not of class C'', provided that*

1. *u is of class C'' in the product complex of Ω' by an r-interval $0 < a \leqq r \leqq b$ by a region in the $(x_{m+1}, \cdots, x_n)$-space.*

2. *The integrals*

$$I_1 = \int_{\Omega'} h'_k(\Omega') u(r, \Omega'; x_{m+1}, \cdots, x_n) d\Omega',$$

$$I_2 = \int_{\Omega'} h_k'(\Omega') \{\partial^2/\partial r^2 + [(m-1)/r](\partial^2/\partial r) + \partial^2/\partial x^2_{m+1} + \cdots + \partial^2/\partial x_n^2\} u d\Omega',$$

$$I_3 = \int_{\Omega'} h'_k(\Omega') \Delta_2 u d\Omega',$$

if improper, converge.

3. *The order of integrations and differentiations in I_2 may be inverted.*

4. *The integral*

$$\int (u\, \partial h'_k/\partial n - h'_k\, \partial u/\partial n) d\Sigma$$

extended over the boundary Σ of a finite part Ω'_f of Ω' approaches zero as Ω'_f expands so as to enclose any inner point of Ω'.

The proof of Theorem V parallels closely the proof of Theorem IV and will be omitted.

It is of interest to point out that Theorem V remains valid if the Ω' integrations are carried out not over all of Ω', but only over part of it, provided that along the additional boundary the integral

$$\int (u\, \partial h'_k/\partial n - h'_k\, \partial u/\partial n) d\Sigma$$

vanishes.

Theorem IV, too, admits of similar extensions to multiple-valued functions h_k, h_k' or to integrations extending over part of the flat or the Riemann surface over which h_k' is single-valued.

8. *Non-Euclidean Spaces.* We shall denote the non-Euclidean space by N_n and consider the analogues of the operators L_k, $L_{k,m}$, $L^*_{k,m}$. Naturally, the Laplacian ∇^2 of E_n is to be replaced by the second differential invariant operator of Beltrami Δ^2 for N_n.

We recall first that the geometry of directions thru a point is the same for N_n as for E_n; therefore we may use θ_i, ω to specify directions thru a point, as in the Euclidean case, without leading to any confusion. The element of length in spherical coördinates r, ω is

$$ds^2 = dr^2 + (\sin cr/c)^2 (ds_\omega)^2,$$

where c is the reciprocal of the space constant and ds_ω is the element of length along a unit sphere in E_n; c^2 is positive for Riemann spaces, negative for Lobachevsky spaces, while the Euclidean case is obtained by letting c^2 approach zero. In terms of these spherical coördinates the operators L_k are defined as in the Euclidean case by means of

$$L_k(u) = h_k(\omega) \int h'_k(\omega') u(r, \omega') d\omega';$$

here h_k, h_k' are the *same* surface spherical harmonics as in the Euclidean case. Likewise the operators $L_{k,m}$ are obtained by applying the m-dimensional case of L_k to m-flats which are totally perpendicular to a fixed $(n - m)$-flat, N_{n-m}:

$$L_{k,m}(u) = h_k(\omega) \int h'_k(\omega') u(r, \omega'; \xi_1, \cdots, \xi_{n-m}) d\omega;$$

here $\xi_1, \cdots, \xi_{n-m}$ are coördinates along N_{n-m}, specifying the center of the subspheres of integration, r is the radius of the subspheres, and ω specifies the direction of the radii; directions thru two centers O_1, O_2, corresponding to the same ω are directions perpendicular to the line O_1O_2 and lying in a half plane bounded by it.

To prove the permutability of $L_{k,m}$ with Δ_2 (the operators L_k result from $L_{k,m}$ if $m = n$), we observe that in terms of the coördinates $r, \omega; \xi_1, \cdots, \xi_{n-m}$ the element of length for N_n is given by

$$ds^2 = dr^2 + \cos^2 cr (ds_\xi)^2 + (\sin^2 cr/c^2)(ds_\omega)^2$$

where $(ds\ )^2$ is the quadratic differential in $d\xi_1, \cdots, d\xi_{n-m}$ which is equal to the square of element of length along N_{n-m}, and $(ds_\omega)^2$, similarly, is equal to

the square of element of length along a Euclidean unit sphere in E_m. Choosing for ξ_i as well as for ω orthogonal coördinates, it is readily verified that

$$\Delta_2 u = \frac{c^2}{\cos^{n-m} cr \sin^{m-1} cr} \frac{\partial}{\partial r}\left[\cos^{n-m} cr \sin^{m-1} cr \frac{\partial u}{\partial r}\right] + \frac{1}{\cos^2 cr} {}_\xi\Delta_2 u + \frac{(cr)^2}{\sin^2 cr} {}_\omega\Delta_2 u,$$

where ${}_\xi\Delta_2$, ${}_\omega\Delta_2$ are the same differential operators as the Beltrami operators for N_{n-m}, and for a Euclidean sphere in E_m, respectively.

We may now show that each of the three parts of Δ_2 is permutable with $L_{k,m}$. Indeed, the permutability of the first two parts follows by changing the order of the ω-integrations and the r- or ξ_i-differentiations. The permutability of the third part follows by applying adaptations of equations (11) and (12) in the form,

$$\int v({}_\omega\Delta_2 u) - u({}_\omega\Delta_2 v)\, d\omega = 0,$$

$${}_\omega\Delta_2(h_k, h_k') = -k(k+m-2)(h_k, h'_k).$$

The proof may be extended to $r = 0$ by means of considerations similar to those used in §§ 3, 5 for the Euclidean cases.†

It will be recalled that in the Euclidean case the operators $L_{k,m}$ are suggested by letting the locus of centers recede to infinity. While no such limiting process may be carried out in the Riemann geometry, for the Lobachevsky geometry the locus of centers N_{n-m} could be placed either at infinity or in the transfinite region. The operators $L^*_{k,m}$ which result involve integrations not along spheres, but along horospheres and equidistant surfaces, respectively. For the transfinite case analogue of $L^*_{k,m}$, it is of interest to note, that the same coördinate system is involved as for $L_{k,n-m}$. The operator in question is

$$L^{**}_{k,m}(u) = h_k(\xi_1, \cdots, \xi_m) \int h'_k(\xi'_1, \cdots, \xi'_m) u(r, \omega; \xi'_1, \cdots, \xi'_m)\, dS_\xi;$$

here the integrations are carried out (not over the loci $r = \text{const.}$, $\xi_i = \text{const.}$

† For L_k an alternative procedure is to map the neighborhood of $r = 0$ on a concentric family of spheres in E_n, radii going into radii of the same length and angles between radii being preserved. The operation $L_k(u)$ for N_n goes into an operation L_k for E_n. As $L_k(u)$ is of class C'' at $r = 0$ for the Euclidean case, it will also be of class C'' for the non-Euclidean case, so that $\Delta_2 L_k(u)$ will exist and be continuous at $r = 0$. Likewise $\Delta_2 u$, continuous at $r = 0$ for N_n, will be continuous for $r = 0$ in E_n; hence $L_k(\Delta_2 u)$ will be continuous at $r = 0$ in E_n and therefore also in N_n.

A similar procedure could also be followed for $L_{k,n}$.

as for $L_{k,n-m}$ but) over $r = \text{const.}$, $\omega = \text{const.}$, that is, along "equidistant" m-dimensional surfaces, a distance r away from a flat N_m; h_k, h'_k are solutions of

$$\Delta_2 h = kh;$$

and dS_ξ is the $(m-1)$-dimensional content of the space $r = 0$. This operator $L^*_{k,m}$ may be shown to be permutable with Δ_2 under conditions analogous to those of Theorem IV.

If the space N_{n-m} of $L_{k,m}$ recedes to infinity in a proper manner, one is led to an element of arc of the form

$$ds^2 = d\xi^2 + e^{2c\xi}(dx_1^2 + \cdots + dx^2_{n-1}),$$

the loci $\xi = \text{const.}$ being equidistant horospheres, and to operators

$$L^*_{k,m}(u) = h_k(x_1, x_m) \int_{-\infty}^{\infty} \cdots \int h'_k(x'_1, \cdots, x'_m)$$
$$\times\, u(\xi; x'_1, \cdots, x'_m; x_{m+1}, \cdots, x_{n-1})\, dx'_1 \cdots dx'_m,$$

where h_k, h'_k are solutions of

$$(\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_m^2)h = kh;$$

the left hand operator is the Beltrami operator along a horosphere $\xi = 0$; $x_{m+1} = \text{const.}, \cdots, x_{n-1} = \text{const.}$ Again the conditions under which one might prove the permutability of $L^*_{k,m}$ and Δ_2 are analogous to those of Theorem IV.

9. *Further extensions; iterated Laplacians.* Because of the very nature of the above theorems the functions u have been assumed to be of class C''. The theory of potential indicates in what direction the above results may be extended to less restricted functions. Thus, for $n = 3$, any function u of class C'' may be considered as the potential due to gravitating matter of volume density $\nabla^2 u/4\pi$ (plus a harmonic function). If one rephrases the various theorems above in terms of potentials and mass distributions one has a formulation which might be applied to such functions u as the potentials due to point, line, and surface distributions, using $\int (\partial u/\partial n)\, d\sum/4\pi$ as the mass inside a surface $\sum$.

Following out these ideas, one arrives at an extension of, say, Theorem II, of the following form: if u is a function, in general of class C', then the function

$$L_k(u) = h_k(\omega) \int h'_k(\omega') u(r, \omega')\, d\omega'$$

is the potential at r, ω obtained by taking the masses whose potential is the function u and replacing each mass initially at r, ω' by a surface distribution of mass over that sphere of density $h_k(\omega)h'_k(\omega')$ per total area of that sphere (or $h_k(\omega)h'_k(\omega')/4\pi$ per unit of the solid angle ω).

If one considers, on the other hand, more regular functions u, say, of class $C^{(2n)}$ it follows by successive applications of the various theorems that the above operators are permutable with iterations of the Laplacian. Thus, for example, if u is of class $C^{(6)}$, then the functions u, $\nabla^2 u$, $\nabla^4 u$ are all of class C''. Hence by applying Theorem II to them, we get

$$\begin{aligned} L_k[\nabla^2(\nabla^4 u)] &= \nabla^2[L_k(\nabla^4 u)], \\ L_k[\nabla^2(\nabla^2 u)] &= \nabla^2[L_k(\nabla^2 u)], \\ L_k[\nabla^2 u] &= \nabla^2[L_k(u)] \quad , \end{aligned}$$

and, transforming each right hand member by means of the equation which follows, arrive at

$$L_k(\nabla^6 u) = \nabla^6[L_k(u)].$$

It will be noted that while the proof outlined does not prove that $L_k(u)$ is of class $C^{(6)}$, it justifies the three fold application of ∇^2 to this function and proves the final result to be a continuous function. That $L_k(u)$ is of class $C^{(6)}$ may be proved by the methods used in § 3.

Returning to the permutability of L_k and ∇^2, it is, of course, trivial to remark that the sum of two operators L_k with the same or different values of the degree k, result in an operator which is permutable with ∇^2; likewise, for the sum of an infinite number of such operators, barring questions of convergence. We shall now show that in a certain sense such a sum constitutes the most general linear functional operator L, permutable with ∇^2 and which replaces a function u over each of a concentric family of spheres by a function depending upon the value of u over that sphere only.

We shall suppose that the operator is of the form

$$L(u) = \int G(r, \omega, \omega')u(r, \omega')d\omega'$$

and that the function G may be expanded above in series of a complete set of surface spherical harmonics in both ω and ω'. Let $h_{k,1}, h_{k,2}, \cdots, h_{k,N_k}$ be a complete set of orthogonalized surface harmonics of degree k, normalized over the unit sphere. Expanding G, we get

$$L(u) = \sum_{k,i} \sum_{k',i'} G_{k,i;k',i'}(r)h_{k,i}(\omega) \int h_{k',i'}(\omega')u(r,\omega')d\omega',$$

assuming term by term integration permissible. Now choose for u the harmonic function $h_{k',i'}(\omega)r^{k'}$ for some fixed values of k', i'. All but one of the integrals vanish. As pointed out at the end of § 1, $L(u)$ must be harmonic if u is harmonic; hence we conclude that

$$\sum_{k,i} G_{k,i;k',i'}(r)r^{k'}h_{k,i}(\omega)$$

is harmonic; therefore the functions $G_{k,i;k',i'}(r)r^{k'}$ must reduce to a constant multiple of r^k:

$$G_{k,i;k',i'}(r) = C_{k,i;k',i'}r^{k-k'}.$$

It remains to show that the constants C vanish for $k \neq k'$.

To prove this choose $u = h_{k',i'}(\omega)r^{-k'-n+2}$. This function is harmonic for $r > 0$; hence $L(u)$ which reduces to

$$\sum_{k,i} C_{k,i;k',i'}r^{k-2k'-n+2}h_k(\omega),$$

must also be harmonic for $r > 0$. This can happen only if C vanishes, or if the exponent of r reduces to $-k-n+2$ or to k. Now the last case can only be realized if $n = 2$, $k' = 0$. The proposition is thus proved except for this special case; that this case forms no exception may now be shown by choosing $u = \log r$ and proceeding as above. In this way one proves that the operator L must reduce to a sum of the operators L_k.

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ON MORSE'S DUALITY RELATIONS FOR MANIFOLDS.*

By Arthur B. Brown.†

1. *Introduction.* Morse has proved ‡ certain duality relations of importance in the calculus of variations. They refer to manifolds with regular boundaries and in one set of relations to immersed sub-complexes, with some analytic restrictions.§ We show that Morse's relations can be derived from certain relations of Lefschetz ¶ involving the homology characters of manifolds, of their closed subsets and of the residual spaces, and from the Poincaré duality theorem. Morse's relations involving immersed sub-complexes are also a corollary of relations of van Kampen,‖ who in addition has results involving intersection numbers, and treats general immersed complexes. By our different methods we obtain some results given neither by Morse nor van Kampen, principally Theorems 6 and 7.

We mention the possibility of extending the results for immersion to the case of an arbitrary closed sub-set of a manifold, for example by the method of infinite complexes, Lefschetz I, Chap. VII.

All results and proofs hold for combinatorial manifolds, and are valid if Betti numbers and homologies are taken either absolute or (mod. p), p any prime greater than unity.

For brevity, we shall generally understand "independent" to mean "linearly independent with respect to homologies".

2. *Manifolds with regular boundaries.*

* Presented to the Society, March 25, 1932.

† Part of the work on this paper was done while the author was a National Research Fellow.

‡ Marston Morse, "The Analysis and Analysis Situs of Regular n-Spreads in $(n+r)$-Space," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 813-817. These relations were brought to my attention by Morse, who suggested that they might possibly be proved by combinatorial processes.

§ We use the terminology of S. Lefschetz's "Topology," *Colloquium Series*, Vol. 12, New York, 1930. (Lefschetz I.)

¶ S. Lefschetz, "Manifolds with a Boundary and Their Transformations," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 429-462. For specific theorems, we shall refer to Lefschetz I.

‖ E. R. van Kampen, "Eine Verallgemeinerung der Alexanderschen Dualitätssatzes," *Amsterdam Proceedings*, Vol. 31 (1928), pp. 899-905. Here, as in Morse's paper, outlines of proofs are given.

THEOREM 1. *Given a connected combinatorial manifold M_n with non-vacuous regular boundary M_{n-1}, let*

$$\rho_i = R_i(M_n) + R_{n-i-1}(M_n) - R_i(M_{n-1}), \qquad (i < n), \tag{2.1}$$

$$\sigma_i = \sum_{k=0}^{i} (-1)^{i-k} \rho_k. \tag{2.2}$$

Then

$$0 \leqq \rho_i, \qquad 0 \leqq \sigma_i, \qquad 0 = \sigma_{n-1}. \tag{2.3}$$

Proof. We apply the definitions and formulas of Lefschetz I, §§ 38 and 39, Chap. III (pp. 149-150), replacing K and L by M_n and M_{n-1} respectively. First we note that $t_p = 0$ for all $p < n$, as follows from its definition and Theorem I on page 153 of Lefschetz I.* From (21) and (22) on page 150 we then obtain

$$R_i(M_n) - R_i(M_{n-1}) + s_i = r_i \qquad (i < n). \tag{2.4}$$

Now from (21) we have

$$s_i = R_{i+1}(M_n;\, M_{n-1}) - r_{i+1}. \tag{2.5}$$

From (9), page 142 of Lefschetz I, we have

$$R_{i+1}(M_n;\, M_{n-1}) = R_{n-i-1}(M_n - M_{n-1}). \tag{2.6}$$

By Theorem II, page 154,

$$R_{n-i-1}(M_n - M_{n-1}) = R_{n-i-1}(M_n). \tag{2.7}$$

Using (2.5), (2.6) and (2.7) we obtain

$$s_i = R_{n-i-1}(M_n) - r_{i+1}. \tag{2.8}$$

Substituting from (2.8) in (2.4) and using (2.1), we have

$$\rho_i = R_i(M_n) + R_{n-i-1}(M_n) - R_i(M_{n-1}) = r_i + r_{i+1}. \tag{2.9}$$

From the hypotheses of the theorem it is evident that $r_0 = r_n = 0$. Using that result together with (2.9), the relations (2.3) become obvious. In fact we have the following result.

THEOREM 2. *The difference in the first inequality of* (2.3) *is* $r_i + r_{i+1}$; *in the second is* $r_i = r_i(M_n - M_{n-1})$.

Theorems 1 and 2 hold for Betti numbers (mod. m), m any integer greater than unity, since the proofs are valid for that case. From the proofs

* Professor Lefschetz has called to my attention that throughout § 44 the words "regular or" should be deleted wherever they occur, as semi-regularity is required.

it is evident that *Theorems* 1 *and* 2 *are valid if* M_{n-1} *is not restricted to be a regular boundary, but is any sub-complex* L *such that* $M_n - L$ *is an open (combinatorial) manifold, and such that* $t_p = 0$ *for all* $p < n$.

3. *Residual complexes.*

THEOREM 3. *Given a connected combinatorial manifold* M_n, *and a sub-complex* K *such that neither* K *nor* $M_n - K$ *is vacuous, then if we set*

$$\rho_i = R_i(K) + R_{n-i-1}(M_n) - R_{n-i-1}(M_n - K) - \delta_0{}^i,$$

$$\sigma_i = \sum_{k=0}^{i} (-1)^{i-k} \rho_k,$$

*the following relations will hold.**

$$(3.1) \quad 0 \leqq \rho_i; \qquad (3.2) \quad 0 \leqq \sigma_i; \qquad (3.3) \quad 0 = \sigma_{n-1}.$$

Proof. First we take the case that K is a combinatorial n-manifold with regular boundary M_{n-1}. Then $B = M_n - K + M_{n-1}$ is likewise an n-manifold with regular boundary M_{n-1}.†

We introduce the following maximal sets of i-cycles, $i = 0, 1, \cdots, n$: A_i, on K, independent on K (as regards homologies) of i-cycles on M_{n-1}; B_i, on B, independent on B of i-cycles on M_{n-1}; $C_i{}^a$, on M_{n-1}, each bounding on K, all independent on B; $C_i{}^b$, on M_{n-1}, each bounding on B, all independent on K; $C_i{}^1$, on M_{n-1}, independent on M_n; $C_i{}^d$, on M_{n-1}, independent on M_{n-1}, each bounding both on K and on B. If we denote by a_i the number of i-cycles in $A_i, \cdots$, the numbers $a_i, b_i, \cdots$, are topological invariants. The following formulas hold.‡

$$(3.4) \qquad R_i(K) = a_i + c_i{}^1 + c_i{}^b.$$

$$(3.5) \qquad R_i(B) = b_i + c_i{}^1 + c_i{}^a.$$

$$(3.6) \qquad R_i(M_{n-1}) = c_i{}^1 + c_i{}^a + c_i{}^b + c_i{}^d.$$

$$(3.7) \qquad R_i(M_n) = a_i + b_i + c_i{}^1 + c^d{}_{i-1}.$$

Using (20), page 149, of Lefschetz I, we have:

$$(3.8) \qquad a_i = a_{n-i}.$$

$$(3.9) \qquad b_i = b_{n-i}.$$

* $\delta_0{}^i = 1$ or 0 according as $i = 0$ or $i \neq 0$.

† Lefschetz I, page 145, Corollary.

‡ A. B. Brown, *American Journal of Mathematics*, Vol. 52 (1930), pp. 251-270; Lemmas 1, 2, 4, 6. The formulas also suggest maximal independent sets of i-cycles for K, B, M_{n-1} and M_n, which are all actually proved to be valid. While the theorems are stated for the absolute or (mod. 2) case, all proofs are valid for the (mod. p) case as well.

Again using the relation $r_i = r_{n-i}$, we rewrite (2.8) in the form $R_i(M_n) - r_i = s_{n-i-1}$ and apply the result to the present K and M_{n-1} in place of M_n and M_{n-1} of § 2, thus obtaining (3.10) below; with B and M_{n-1} we obtain (3.11). We omit any proof beyond referring to (3.4), (3.5), (3.6) and the definitions above.

$$c_i^b + c_i^1 = c^a_{n-i-1} + c^d_{n-i-1}. \tag{3.10}$$

$$c_i^a + c_i^1 = c^b_{n-i-1} + c^d_{n-i-1}. \tag{3.11}$$

In the following, (3.12) follows from (3.6) and the Poincaré duality theorem for M_{n-1}; (3.13) from (3.10), (3.11) and (3.12); (3.14) from (3.11), (3.11) with i replaced by $n-i-1$, and (3.13).

$$c_i^1 + c_i^a + c_i^b + c_i^d = c^1_{n-i-1} + c^a_{n-i-1} + c^b_{n-i-1} + c^d_{n-i-1}. \tag{3.12}$$

$$c_i^1 + c^1_{n-i-1} = c_i^d + c^d_{n-i-1}. \tag{3.13}$$

$$c_i^a + c^a_{n-i-1} = c_i^b + c^b_{n-i-1}. \tag{3.14}$$

By a duality relation of Lefschetz we have (cf. (2.6))

$$R_i(K - M_{n-1}) = R_{n-i}(K;\, M_{n-1}). \tag{3.15}$$

This gives us the relation

$$a_i + c_i^1 + c_i^b = a_{n-i} + c^d_{n-i-1} + c^a_{n-i-1}. \tag{3.16}$$

That the left-hand sides of (3.15) and (3.16) are equal follows from (3.4) and the relation $R_i(K - M_{n-1}) = R_i(K)$. (Cf. (2.7)). The proof that the right-hand sides are equal is easily given by use of (21) on page 150 of Lefschetz I, using the fact that $t_i = 0$ (cf. § 2).

The following equations are derived as explained below.

$$c_i^1 + c^d_{i-1} = c^1_{n-i} + c^d_{n-i-1}. \tag{3.17}$$

$$c_i^d + c^d_{i-1} = c^1_{n-i} + c^1_{n-i-1}. \tag{3.18}$$

$$c_0^d = c^1_{n-1}. \tag{3.19}$$

$$c_i^1 = c^d_{n-i-1}, \qquad (i = 0, 1, \cdots, n-1). \tag{3.20}$$

$$c_i^a = c^b_{n-i-1}, \qquad (i = 0, 1, \cdots, n-1). \tag{3.21}$$

Here (3.17) is obtained from the Poincaré duality relation for M_n, with the use of (3.7), (3.8) and (3.9); (3.18) from (3.13) and (3.17) by subtraction; (3.19) from (3.18) by setting $i = 0$ and observing that $c^d_{-1} = c_n^1 = 0$, as follows from the facts that there are no cycles of dimension

-1, and that M_{n-1} is of dimension only $n-1$; (3.20) from (3.18) and (3.19) by a succession of simple steps of algebra; (3.21) by substituting from (3.8) and (3.20) in (3.16), and replacing i by $n-i-1$.

We can now establish (3.1), (3.2) and (3.3). First we substitute from (3.21), with i replaced by $n-i-1$, in (3.4); then from (3.20) with i replaced by $n-i-1$, and from (3.8), in (3.7), to obtain the following relations.

$$R_i(K) = a_i + c^a{}_{n-i-1} + c_i{}^1. \tag{3.22}$$

$$R_i(M_n) = a_{n-i} + b_i + c_i{}^1 + c^1{}_{n-i}. \tag{3.23}$$

Since B is an n-manifold with regular boundary, $R_i(B) = R_i(M_n - K)$. We also have the obvious relations $a_0 = a_n = c_n{}^1 = 0$, $c_0{}^1 = 1$. By using these relations and (3.5), (3.22) and (3.23), we find upon substitution that (3.1) would be an equality if $a_i + a_{i+1} + c_i{}^1 + c^1{}_{i+1} - \delta_0{}^i$ were added to its left-hand side; that (3.2) would be an equality if $a_{i+1} + c^1{}_{i+1}$ were added to its left-hand side; and that (3.3) is valid. As $c_i{}^1 \geqq \delta_0{}^i$, Theorem 3 is therefore proved for the case that M_n is an n-manifold with regular boundary.

We consider now the case that K is any sub-complex of M_n. We denote by R' and R'' the first and second regular subdivisions, respectively, of any complex R. Let N' be the M'_n-neighborhood of K', less K'. (Thus $N' \not> K'$.) Let C be the complex which is the point-set boundary of the D''-neighborhood of K''. Then $N' + K'$ is a normal neighborhood of K', and N' is covered by a field F of curves, where each curve has its end-points on the boundaries on $N' + K'$ and of K' respectively, and cuts C in one point.* Hence N' is homeomorphic to the product P of C by a 1-cell. As N' is open on M'_n and M'_n is an n-manifold, N' is an open n-manifold; hence, by the topological invariance of the combinatorial manifold,† P is an open n-manifold. Therefore, if E_i is an i-cell of P, $LK_P(E_i)$ is an H_{n-i-1} (Lefschetz I, page 120, condition (b)). But if E'_{i-1} is the $(i-1)$-cell of C which generates E_i, then $LK_C(E'_{i-1})$ has the same structure as $LK_P(E_i)$, hence is likewise an $(n-i-1)$-sphere. Hence C, composed of simplicial cells, is an $(n-1)$-manifold. Consequently, if $K_1 = K'' + N'' + C$ and $B_1 = M_n'' - K_1$, then K_1 and $B_1 + C$ are n-manifolds each with regular boundary C.‡

From the proof for the case first considered it then follows that the relations of Theorem 3 are valid for A_1 and B_1 in place of K and $M_n - K$.

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* Cf. Lefschetz I, page 91.

† E. R. van Kampen, "Die kombinatorische Topologie und die Dualitätssatze," *Leyden thesis* (1929), The Hague. Cf. Lefschetz I, page 155, § 4, Theorem.

‡ Lefschetz I, page 145, Corollary.

Now by use of the field F of curves in N' it is easily shown that K_1 and B_1 have the same Betti numbers as K and $M_n - K$, respectively.* As a consequence, the relations of Theorem 3 are valid for the given K, and the proof is complete.

We now state five theorems obtained (see below) from the following relations: Theorem 4 from the formulas for the differences in the sides of (3.1) and (3.2), mentioned at the end of the proof above for the case that K is a manifold with regular boundary; Theorem 5 from the same results, by symmetry; Theorems 6, 7, 8 from (3.20), (3.8) and (3.21) respectively. We first remark that Theorem 4 and the last part of Theorem 5 are not equivalent, since K is closed and $M_n - K$ is open, on M_n. The proofs of the theorems consist in demonstrating them, by use of the above-mentioned formulas, for the case that K is an n-manifold with regular boundary, and then extending the results to the case of a sub-complex, by use of C, N', etc., occurring at the end of the last proof. We shall omit the detailed proofs.

For brevity in stating the theorems, we introduce the following notation, where $R \supset L$: for the number of i-cycles in a maximal set —— on L, independent (with regard to homologies) on R, $\rho_i(L, R)$; on L, independent on R, but each $\sim$ an i-cycle on $R - L$, $\sigma_i(L, R)$; on R, independent of the i-cycles on L and those on $R - L$, $\tau_i(R, L)$; on L, independent on R of the i-cycles on $R - L$, $\mu_i(L, R)$; on L, independent on L, each bounding on R, $\nu_i(L, R)$.

THEOREM 4. *The difference between the two sides in* (3.1) *is* $\rho_i(K, M_n) + \rho_{i+1}(K, M_n) - \delta_0{}^i$, *and the difference for* (3.2) *is* $\rho_{i+1}(K, M_n)$.

THEOREM 5. *Under the hypotheses of Theorem* 3, *relations* (3.1), (3.2), (3.3) *remain valid if every* $R_j(K)$ *is replaced by* $R_j(M_n - K)$ *and every* $R_j(M_n - K)$ *by* $R_j(M_n)$. *The difference between the two sides in the relation corresponding to* (3.1) *is then* $\rho_i(M_n - K, M_n) + \rho_{i+1}(M_n - K, M_n) - \delta_0{}^i$, *and the difference for the relation corresponding to* (3.2) *is* $\rho_{i+1}(M_n - K, M_n)$.

THEOREM 6. *If M_n is a combinatorial n-manifold, and K a sub-complex, then* $\sigma_i(K, M_n) = \tau_{n-i}(M_n, K)$.

THEOREM 7. *Under the hypotheses of Theorem* 6,

$$\mu_i(K, M) = \mu_{n-i}(K, M_n).\dagger$$

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* The result may also be obtained simply by use of the author's Lemma A, *Annals of Mathematics*, Vol. 32 (1931), p. 514.

† Theorems 6 and 7 together give van Kampen's relation 3°, for the present case, insofar as the number of cycles in each set is involved.

THEOREM 8. *Under the hypotheses of Theorem 6,*

$$v_i(K, M_n) = v_{n-i-1}(M_n - K, M_n).^*$$

4. *Manifolds with the Betti numbers of an n-sphere.* If M_n has the Betti numbers of an n-sphere,† the quantities appearing in Theorems 4, 5, 7 become all zero, and those in Theorem 6 equal $\delta_0{}^i$. Hence Theorems 6 and 7 become trivial, and Theorems 3, 4, 5 and 8 reduce to the Alexander duality theorem for sub-complexes. But Theorem 1 gives an interesting result (of Morse), when K is immersed in ‡ M_n, as follows.

THEOREM 9. *Let K be a combinatorial n-manifold with non-vacuous regular boundary M_{n-1}, and M_n a combinatorial n-manifold with the Betti numbers of an n-sphere. If K can be immersed in M_n, then the following relations are valid:*

$$R_i(M_{n-1}) = R_i(K) + R_{n-i-1}(K). \tag{4.1}$$

Proof. Since (4.1) for K would be a consequence of (4.1) for the separate connected parts of K, obtained by summing, it will be sufficient to prove (4.1) for the case that K is connected. Then the hypotheses of Theorem 1 are satisfied, and, according to Theorem 2, (4.1) will be proved if we show that any i-cycle on K is $\approx$ on K to an i-cycle on M_{n-1}.

Since K is not in general a sub-complex of M_n, the result is not obvious. However, by using the data on the Betti numbers of M_n, and a field of curves in a normal neighborhood of M_{n-1} *on* K (obtained by regular subdivision; cf. the last part of the proof of Theorem 3), the proof is easily carried through, and we shall not give the details.

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* Theorem 8 corresponds to van Kampen's 1°, for our case, without, of course, giving his results regarding looping.

† This does not necessarily make M_n a combinatorial n-sphere, as no condition is imposed on the torsion coefficients of D.

‡ "Immersed in" means "coincident, as a result of a homeomorphism, with a sub-set of."

INVARIANTS OF INTERSECTION OF TWO CURVES ON A SURFACE.

By Ernest P. Lane.

1. *Introduction.* In a recent note * Bompiani has developed some stimulating ideas relative to certain invariants of intersection of two skew curves in ordinary space. He shows, in the first instance, that if two curves C, C' pass through a point P with distinct tangents t, t' at P and also with distinct osculating planes at P, whose line of intersection is different from t and from t', then there exist, through P and in the plane determined by t, t', two lines which are *principal* in the sense that the two cones projecting C, C' from any point on either line have contact of the second or higher order along their common generator through P, instead of contact of the first order as would ordinarily be the case if the center of projection were chosen at random in the plane of t, t'. Bompiani shows that the two principal lines separate the tangents t, t' harmonically. Moreover, he shows that on each principal line there is a point which is *principal* in the sense that if the projecting cones have their vertices at this point, the cones have contact of the third order.

In the case that the tangents t, t' are distinct but the osculating planes are coincident, Bompiani shows the existence of *three principal lines*, and of a principal point on each of these lines. He applies these considerations to the interesting case in which the two curves are asymptotic curves on a surface, and deduces some remarkable results which need not be reported in entirety here. But we shall refer to one of these results later on in Section 4.

The present note had its inception with the idea of applying the considerations summarized in the opening paragraph above to the case in which the two curves belong respectively to the two families of a conjugate net of curves on a surface. Some interesting results were found which will be presented forthwith. Moreover, some kindred considerations concerning certain quadric surfaces containing the asymptotic tangents at a point of a surface will be adjoined.

2. *Principal Lines.* We proceed to indicate an analytic basis for the

* Bompiani, "Invarianti d'intersezione di due curve sghembe," *Rendiconti dei Lincei*, Ser. 6, Vol. 14 (1931), pp. 456-461.

study of a surface referred to a conjugate net, and to determine the two principal lines, at a point of the surface, of the two curves of the net that pass through the point.

If the projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$ of a point P_x in ordinary space are given as analytic functions of two (and not fewer) independent variables u, v by equations of the form

$$x = x(u, v), \tag{1}$$

the locus of P_x as u, v vary is a proper analytic surface S. If the parametric curves on S form a conjugate net, the four coördinates x and the four coördinates y of a point on *the axis* of the point P_x satisfy a completely integrable system of linear partial differential equations of the form *

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny \qquad (LN \neq 0). \end{aligned} \tag{2}$$

The ray-points, or Laplace transformed points, x_{-1}, x_1 of the point P_x are given by the formulas

$$x_{-1} = x_u - bx, \qquad x_1 = x_v - ax. \tag{3}$$

The expansion for the u-curve at the point P_x of the form used by Bompiani in his note already cited may be calculated in the following way. The coördinates X of a point near P_x and on the u-curve through P_x are expressible by Taylor's expansion as power series in the increment Δu corresponding to displacement from P_x to the point X along the u-curve,

$$X = x + x_u \, \Delta u + x_{uu} \, \Delta u^2/2 + x_{uuu} \, \Delta u^3/6 + \cdots.$$

Expressing each of x_{uu}, x_{uuu} as a linear combination of x, x_{-1}, x_1, y we find

$$X = y_1 x + y_2 x_{-1} + y_3 x_1 + y_4 y,$$

where the local coördinates $y_1, \cdots, y_4$ of the point X are given by the expansions

$$\begin{aligned} y_1 &= 1 + b\,\Delta u + (p + b\alpha)\Delta u^2/2 + \cdots, \\ y_2 &= \Delta u + \alpha\,\Delta u^2/2 + \cdots, \\ y_3 &= \boldsymbol{H}\Delta u^2/6r + \cdots, \\ y_4 &= L\,\Delta u^2/2 + L[\alpha + b - (\log r)_u]\,\Delta u^3/6 + \cdots, \end{aligned} \tag{4}$$

and we have placed

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* Lane, ***Projective Differential Geometry of Curves and Surfaces***, **University of Chicago Press (1932), p. 138.**

$$(5) \qquad r = N/L, \qquad \boldsymbol{H} = b_v + ab + c - \delta_u.$$

Introducing non-homogeneous coördinates by the definitions

$$(6) \qquad x = y_2/y_1, \qquad y = y_3/y_1, \qquad z = y_4/y_1,$$

we find

$$(7) \qquad \begin{aligned} x &= \Delta u + (\alpha - 2b)\Delta u^2/2 + \cdots, \\ y &= \boldsymbol{H}\,\Delta u^3/6r + \cdots, \\ z &= L\,\Delta u^2/2 + L[\alpha - 2b - (\log r)_u]\,\Delta u^3/6 + \cdots. \end{aligned}$$

Inverting the first of these series we obtain

$$\Delta u = x - (\alpha - 2b)x^2/2 + \cdots,$$

and substituting this series for Δu in the last two of the series (7) we arrive at the required expansions for the u-curve, namely,

$$(8) \qquad y = \boldsymbol{H}x^3/6r + \cdots, \qquad z = Lx^2/2 + 4L\mathfrak{C}'x^3/3 + \cdots,$$

where we have placed

$$(9) \qquad 8\mathfrak{C}' = 4b - 2\alpha - (\log r)_u.$$

Similar calculations lead to the following expansions for the v-curve at the point P_x,

$$(10) \qquad x = r\boldsymbol{K}y^3/6 + \cdots, \qquad z = Ny^2/2 + 4N\mathfrak{B}'y^3/3 + \cdots,$$

where we have placed

$$(11) \qquad \boldsymbol{K} = a_u + ab + c - \alpha_v, \qquad 8\mathfrak{B}' = 4a - 2\delta + (\log r)_v.$$

Without further calculations we may compare the expansions (8), (10) with equations (1) on page 457 of Bompiani's note. Then making use of equations (5) on page 458 of Bompiani's note for the principal lines we reach immediately the following result:

The equations of the principal lines of the parametric curves at the point P_x of the surface S are

$$(12) \qquad z = Lx^2 - Ny^2 = 0.$$

The associate conjugate net of the parametric net on the surface S is by definition the unique conjugate net whose tangents (called *associate conjugate tangents*) at each point of S separate harmonically the parametric tangents at the point. The curvilinear differential equation of the associate conjugate net is

$$(13) \qquad L\,du^2 - N\,dv^2 = 0.$$

Consequently we have the theorem:

The principal lines, at a point of a conjugate net, of the two curves of the net that pass through the point are the associate conjugate tangents at the point.

3. *Principal Points and Principal Join.* We next determine the two principal points of the two curves of the parametric conjugate net at the point P_x, and study the line joining them.

Making use of equation (6) on page 458 of Bompiani's note we find that *the equation of the line joining the principal points is*

$$8(\mathfrak{C}'x + \mathfrak{B}'y) = 3. \tag{14}$$

We propose to call this line *the principal join* of the fundamental parametric curves at the point P_x. Solving equations (12) and (14) simultaneously one may obtain the non-homogeneous local coördinates of the principal points. Then passing from non-homogeneous local coördinates to general homogeneous coördinates one finds that *the points*

$$x_{-1} + 8\mathfrak{C}'x/3 \pm (x_1 + 8\mathfrak{B}'x/3)/r^{1/2} \tag{15}$$

are the principal points of the parametric curves at the point P_x.

Adding and subtracting the two formulas (15), one taken with the plus sign and one with the minus sign, we obtain the result:

The principal join crosses the parametric tangents at the point P_x in the points

$$x_{-1} + 8\mathfrak{C}'x/3, \qquad x_1 + 8\mathfrak{B}'x/3. \tag{16}$$

Recalling that a net is *quadratic*, i. e., lies on a quadric surface, in case

$$\mathfrak{B}' = \mathfrak{C}' = 0,$$

and further recalling that *the ray* of the parametric net crosses the parametric tangents at P_x in the ray-points x_{-1}, x_1, we have the theorem:

The principal join coincides with the ray at each point of a conjugate net if, and only if, the net is quadratic.

We shall suppose hereinafter that the fundamental conjugate net is not quadratic. It is not difficult to calculate the differential equation of the curves corresponding on the surface S to the developables of the congruence of principal joins, and to determine the focal surfaces of this congruence, but we shall have no immediate use for these results and so refrain from including them here.

The associate ray-points, i. e., the ray-points of the associate conjugate net, at the point P_x are easily shown to be given by the formulas

$$x_{-1} + 2\mathfrak{C}'x \pm (x_1 + 2\mathfrak{B}'x)/r^{1/2}. \tag{17}$$

Consequently *the associate ray*, which passes through these two points, crosses the parametric tangents at P_x in the points

$$x_{-1} + 2\mathfrak{C}'x, \qquad x_1 + 2\mathfrak{B}'x. \tag{18}$$

Taking suitable linear combinations of the formulas (16), (18) we may demonstrate the following theorem:

The ray, the associate ray, and the principal join at a point x of the parametric conjugate net are concurrent in the point z defined by

$$z = \mathfrak{B}'x_{-1} - \mathfrak{C}'x_1. \tag{19}$$

Moreover, a brief calculation, which we shall omit, suffices to demonstrate the following theorem:

The cross ratio of the ray, the associate ray, the line xz, and the principal join in the order named is 1/4.

4. *Studies in Asymptotic Parameters.* For many purposes, when studying conjugate nets on a surface S, it is convenient to take the asymptotic curves on the surface as parametric. In this section we shall make use of Fubini's canonical form of the differential equations of the surface, supposed not ruled and referred to its asymptotic curves, namely,

$$x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v \quad (\theta = \log \beta\gamma). \tag{20}$$

The notation in this section will be that employed in Chapter III of the author's forthcoming book previously cited.

The curvilinear differential equation of any conjugate net N_λ on the surface S can now be written in the form

$$dv^2 - \lambda^2\, du^2 = 0. \tag{21}$$

The equation of the associate conjugate net is obtained by changing the sign of λ^2.

By *a line* l_1 we mean any line through a general point P_x of the surface S and not lying in the tangent plane of S at P_x. Such a line joins the point x to the point y defined by placing

$$y = -ax_u - bx_v + x_{uv}, \tag{22}$$

where a, b are functions of u, v. Dually, by *a line* l_2 we mean any line in

the tangent plane of the surface S at the point P_x but not passing through P_x. Such a line joins the points ρ, σ defined by placing

$$\rho = x_u - bx, \qquad \sigma = x_v - ax, \tag{23}$$

where a, b are functions of u, v. When the functions a, b are the same in equations (23) as in (22), the lines l_1, l_2 are commonly called *reciprocal lines,* because they are reciprocal polar lines with respect to the quadric of Lie, whose equation referred to the tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point is

$$2(x_2x_3 - x_1x_4) - (\beta\gamma + \theta_{uv})x_4^2 = 0. \tag{24}$$

It is known that *the ray* of the net N_λ is a line l_2, for which the functions a, b in (23) are given by

$$2a = \theta_v + (\log\lambda)_v - \beta/\lambda^2, \qquad 2b = \theta_u - (\log\lambda)_u - \gamma\lambda^2, \tag{25}$$

while for *the associate ray* it is only necessary to change the sign of λ^2. Moreover, it is known that *the flex-ray* of the net N_λ, i. e., the line of inflexions of the ray-point cubic of the pencil of conjugate nets determined by the net N_λ, is a line l_2 for which the functions a, b are given by

$$2a = \theta_v + (\log\lambda)_v, \qquad 2b = \theta_u - (\log\lambda)_u. \tag{26}$$

It is now easy to verify the conclusion:

The ray, associate ray, flex-ray, and principal join at the point P_x of the net N_λ are concurrent in the point z defined by placing

$$z = (\beta/\lambda^2)\{x_u - [\theta_u - (\log\lambda)_u]x/2\} - \gamma\lambda^2\{x_v - [\theta_v + (\log\lambda)_v]x/2\}. \tag{27}$$

In order to calculate *the functions a, b for the principal join* we make use of the concluding theorem of Section 3, and thus find

$$2a = \theta_v + (\log\lambda)_v + 5\beta/3\lambda^2, \qquad 2b = \theta_u - (\log\lambda)_u + 5\gamma\lambda^2/3. \tag{28}$$

By means of this result it is easy to verify the truth of the following statement:

The cross ratio of the ray, associate ray, flex-ray, and principal join in the order named at any point P_x of a net N_λ is $-1/4$.

The relation between a conjugate net and its associate conjugate net is obviously entirely reciprocal. So we are led to define *the associate principal join* at a point of a net N_λ, i. e., the principal join of the associate conjugate net of N_λ. For the associate principal join the functions a, b are obviously

obtained from the formulas (28) by merely changing the sign of λ^2. The associate principal join passes through the point z defined by (27). Moreover, it turns out that the cross ratio of the principal join, associate principal join, flex-ray, and line xz in the order named is -1. Thus we prove the theorem:

At a point x of a conjugate net N_λ, the principal join and associate principal join separate harmonically the flex-ray and the line xz connecting the point x to the point z of concurrency of the principal join and associate principal join.

We conclude these studies with some comments on certain non-singular quadric surfaces containing the asymptotic tangents at a point P_x of a surface S. The equation of any such quadric, referred to the tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point, can be written in the form

$$x_2x_3 + x_4(k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4) = 0, \tag{29}$$

where $k_1, \cdots, k_4$ are arbitrary. A line l_1 has the same polar line l_2 with respect to such a quadric as with respect to the quadric of Lie (24) if, and only if, the coefficients k_1, k_2, k_3 are connected by the relations

$$k_2 = b(1 + k_1), \qquad k_3 = a(1 + k_1). \tag{30}$$

Moreover, any quadric that has second-order contact with the surface S at the point P_x has an equation of the form (29) with $k_1 = -1$. Finally, the quadrics (29) for which $k_1 = -1$ and $k_2 = k_3 = 0$ are commonly called the quadrics of Darboux. Inspection of the relations (30) makes evident the following proposition:

If a quadric has contact of the second order at a point P_x of a surface S and if the polar relation of any two lines l_1, l_2 with respect to this quadric is equivalent to the polar relation of the lines with respect to the quadric of Lie at P_x, then the quadric is a quadric of Darboux of S at P_x.

It is further known that the quadrics (29) for which $k_1 = -3$ are the quadrics having contact of the third order with the asymptotic curves at the point P_x. Moreover, if also

$$k_2 = -\phi/2, \qquad k_3 = -\psi/2, \tag{31}$$

where

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v, \tag{32}$$

it is known that a quadric (29) has contact of the fourth order with the asymptotic curves at P_x. Thus we see that *any quadric of the pencil*

$$x_2x_3 + x_4(-3x_1 - \phi x_2/2 - \psi x_3/2 + k_4x_4) = 0 \tag{33}$$

has contact of the fourth order with the asymptotic curves at the point P_x of the surface S. On page 461 of Bompiani's note previously cited he erroneously concludes that the pencil of quadrics having this property is the pencil of quadrics of Darboux. It is now proposed to name the pencil of quadrics (33) at the point P_x of the surface S *the principal pencil* of quadrics of S at P_x; any one of these quadrics may be called a *principal quadric.* If a unique covariant quadric of this pencil is desired, we may choose the one that passes through the covariant point $(0, 0, 0, 1)$. For this quadric we have $k_4 = 0$.

The polar line of the projective normal, joining the points $(1, 0, 0, 0)$, $(0, 0, 0, 1)$, with respect to any one of the principal quadrics (33) is the line l_2 for which

$$a = \psi/6, \qquad b = \phi/6. \tag{34}$$

The reciprocal polar line l_1 of this line l_2 with respect to the quadric of Lie is the canonical line called *the first principal line* at the point P_x. Thus a new characterization of this line is discovered, which should be compared with that given * by Fubini and Čech, and which may be stated in the following words:

The first principal line l_1 is the polar line with respect to any quadric of Darboux of that line l_2 which is the polar line of the projective normal with respect to any principal quadric.

Finally let us undertake to find a pair of lines l_1, l_2 which are reciprocal polars with respect to the quadric of Lie (or any quadric of Darboux) and are also reciprocal polars with respect to any principal quadric (33). Starting with any line l_1 we find its polar line with respect to any quadric of Darboux and also its polar line with respect to any quadric (33). Demanding that the latter two lines shall coincide we find that they coincide in the second canonical edge of Green, for which

$$a = \psi/4, \qquad b = \phi/4. \tag{35}$$

Thus we obtain a new characterization of the canonical edges of Green which may be formulated in the following theorem:

The canonical edges of Green are the only pair of lines l_1, l_2 which are reciprocal polars both with respect to the quadrics of Darboux and with respect to the principal quadrics.

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* Fubini and Čech, *Geometria Proiettiva Differenziale*, Zanichelli, Vol. 1 (1926), p. 143.

SYSTEMS OF INVOLUTORIAL BIRATIONAL TRANSFORMATIONS CONTAINED MULTIPLY IN SPECIAL LINEAR LINE COMPLEXES.

By Evelyn Teresa Carroll.

Introduction. Birational transformations in space belonging to special line complexes have been treated by Pieri * and by H. A. Davis †; incidental mention of the subject has been made by various other authors. Pieri's method is synthetic, and in general no details have been given, while Davis has considered cases not treated here. The present paper contains a transformation not heretofore discussed and involves an interesting involution of fundamental elements along the basis line.

1. *Definition of the transformation.* Given a line $d \equiv x_1 = 0,\ x_2 = 0$ and a pencil of surfaces of order n, containing d to multiplicity $n-2$

$$\lambda_2 F_n(x) - \lambda_1 F'_n(x) = 0, \tag{1}$$

wherein

$$F_n(x) \equiv \sum_{\mu=0}^{n-2} x_1^{n-\mu-2} x_2^{\mu} u_\mu(x)$$

and

$$F'_n(x) \equiv \sum_{\mu=0}^{n-2} x_1^{n-\mu-2} x_2^{\mu} v_\mu(x),$$

$u_\mu(x)$ and $v_\mu(x)$ being quadratic forms in (x_1, x_2, x_3, x_4).

Let $(z) \equiv (0, 0, z_3, z_4)$ be a variable point on the line d, and let the pencil of surfaces (1) be connected with it by the relation

$$z_3\phi_1(\lambda_1, \lambda_2) - z_4\phi_2(\lambda_1, \lambda_2) = 0, \tag{2}$$

where ϕ_i $(i = 1, 2)$ is a binary form of order k.

A point (y) of space determines a surface of the pencil (1); so that (1) may be wrtten

$$F'_n(y)F_n(x) - F_n(y)F'_n(x) = 0 \tag{3}$$

and $z_3 = \phi_2[F_n(y), F'_n(y)]$ and $z_4 = \phi_1[F_n(y), F'_n(y)]$.

* M. Pieri, "Sulle trasformazioni birazionali dello spazio inerenti a un complesso lineare speciale," *Rendiconti del Circolo Matematico di Palermo*, Vol. 6 (1892), pp. 234-244.

† H. A. Davis, "Involutorial Transformations Belonging to a Linear Line Complex," *American Journal of Mathematics*, Vol. 52 (1930), pp. 53-71.

A point (x) on the line joining (y) to (z) has coördinates of the form

(4) $\rho x_1 = \tau y_1, \quad \rho x_2 = \tau y_2, \quad \rho x_3 = \tau y_3 + \sigma\phi_2(F), \quad \rho x_4 = \tau y_4 + \sigma\phi_1(F).$

The residual point (y') in which this line meets (3) again may be determined by applying the equations of transformation (4) to (3). The result in its simplified form is

$$\sigma[F'_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} u_\mu(\phi) - F_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} v_\mu(\phi)]$$
$$+ 2\tau[F'_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} u_\mu(y, \phi) - F_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} v_\mu(y, \phi)] = 0,$$

in which $u_\mu(\phi)$ and $v_\mu(\phi)$ are quadratic forms in

$$[0, 0, \phi_2(F_n(y), F'_n(y)), \phi_1(F_n(y), F'_n(y))]$$

and $u_\mu(y, \phi)$ and $v_\mu(y, \phi)$ are the polar forms of

$$(z) \equiv [0, 0, \phi_2(F_n(y), F'_n(y)), \phi_1(F_n(y), F'_n(y))]$$

in regard to $u_\mu(y)$ and $v_\mu(y)$, respectively. Therefore

$$\sigma \equiv -2[F'_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} u_\mu(y, \phi) - F_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} v_\mu(y, \phi)]$$
$$\tau \equiv F'_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} u_\mu(\phi) - F_n(y) \sum_{\mu=0}^{n-2} y_1^{n-\mu-2} y_2^{\mu} v_\mu(\phi).$$

The transformation is of order $2n(k+1) - 1$, where τ and σ have the orders $2[n(k+1) - 1]$ and $n(k+2) - 1$, respectively. The line d appears to multiplicity $(n-2)(k+2)$ on $\sigma = 0$ and to multiplicity $2(n-2)(k+1)$ on $\tau = 0$ and on the surfaces which are conjugates of planes in the transformation.

The image of the line d is the surface $\tau = 0$; $\sigma = 0$ is the locus of invariant points.

2. *Fundamental basis curve.* The surfaces $F_n(x) = 0$ and $F'_n(x) = 0$ intersect in a curve of order n^2, composed of d to multiplicity $n-2$ on each surface and a curve $\gamma_{4(n-1)}$ of order $4(n-1)$. The latter is $k+1$ fold on $\sigma = 0$; it is $2k+1$ fold on $\tau = 0$, and on the surfaces conjugate to the planes of space.

If a plane is passed through d, it will intersect $F_n(x)$ in a curve of order n, which consists of d taken $(n-2)$ times and a conic. This is likewise true for $F'_n(k)$ and the two conics intersect in four points. As a plane intersects $\gamma_{4(n-1)}$ in $4(n-1)$ points, $\gamma_{4(n-1)}$ and d must intersect in $4(n-2)$ points.

3. *Plane transformation in planes through d.* Since every plane through d is transformed into itself, it is expedient to discuss the plane transformation. The plane meets $\gamma_{4(n-1)}$ in four points, A, B, C, and D, the basis of a pencil of conics.

In the plane this pencil of conics may be represented by the equation

$$\lambda_2 C_1(x) - \lambda_1 C_2(x) = 0, \tag{5}$$

where $C_1(x) \equiv \sum_1^3 a_{ij}x_ix_j$ and $C_2(x) \equiv \sum_1^3 b_{ij}x_ix_j$, and the line d may be considered to have the equation $x_3 = 0$. Let the pencil of conics and the point $(z) \equiv (z_1, z_2, 0)$ be connected by the relation

$$z_1\phi_2(\lambda_1, \lambda_2) - z_2\phi_1(\lambda_1, \lambda_2) = 0, \tag{6}$$

where ϕ_i $(i = 1, 2)$ is a binary form of order k.

A point (y) in the plane determines a conic of the pencil; hence

$$C_2(y)C_1(x) - C_1(y)C_2(x) = 0, \tag{7}$$

and

$$z_1 = \phi_1[C_1(y), C_2(y)], \qquad z_2 = \phi_2[C_1(y), C_2(y)]. \tag{8}$$

A point (x) on the line joining (y) to (z) has coördinates of the form

$$x_1 = \tau y_1 + \sigma\phi_1(C), \qquad x_2 = \tau y_2 + \sigma\phi_2(C), \qquad x_3 = \tau y_3. \tag{9}$$

When these equations of transformation have been applied to (7) and the necessary expansion has been performed, the resulting form of the equation is

$$\sigma[C_1(\phi)C_2(y) - C_2(\phi)C_1(y)] + 2\tau[C_1(\phi, y)C_2(y) - C_2(\phi, y)C_1(y)] = 0$$

in which $C_i(\phi)$ $(i = 1, 2)$ are quadratic forms in $(\phi_1, \phi_2, 0)$ and $C_i(\phi, y)$ are the polar forms of $(\phi_1, \phi_2, 0)$ with respect to $C_i(y)$. Therefore

$$\sigma = -2[C_1(\phi, y)C_2(y) - C_2(\phi, y)C_1(y)]$$
$$\tau = C_1(\phi)C_2(y) - C_2(\phi)C_1(y).$$

The order of this transformation is $4k + 3$, while σ and τ have orders $2k + 3$ and $2(2k + 1)$, respectively. The points A, B, C, and D occur to multiplicity $2k + 1$ on the curve $\tau = 0$ and on the general curves, conjugates of the straight lines of the plane; and to multiplicity $k + 1$ on the curve $\sigma = 0$, which is the locus of invariant points.

The class of the involution is k, as any line contains k pairs of conjugates; a straight line is transformed into a curve of order $4k + 3$, which meets the

given line in $4k+3$ points, $2k+3$ of which are fixed, the remaining $2k$ points being arranged in k pairs of conjugates.

$\tau = 0$ is composite, being composed of $2k+1$ conics of the pencil. Every point in which any one of these $2k+1$ conics meets $\sigma = 0$ is a fundamental point. The number of such intersections is $4k+6$, $4(k+1)$ of which fall at the base points; this leaves two points of each conic on $\sigma = 0$ and they may be distinct, consecutive, or coincident. No additional fundamental points exist as can be seen from the equations of the involution.

Since the involution is a Cremona transformation, the two fundamental equations $\Sigma\, \alpha_i\, i = 3(n-1)$ and $\Sigma\, \alpha_i\, i^2 = n^2 - 1$ must be satisfied. Here $n = 4k+3$ and $\alpha_{2k+1} = 4$; either $\alpha_1 = 4k+2$ or $\alpha_2 = 2k+1$ and all other α_i's equal to zero satisfy the first equation; the second alternative satisfies the second equation.

Hence *

$$C_{4k+3} : 4^{2k+1}(2k+1)^2.$$

There are $2k+1$ fundamental conics belonging to the given pencil and each passing through one of the double points, and four curves of order $2k+1$, images of A, B, C, and D, respectively; these latter curves have one of the base points to multiplicity $k+1$ and each of the others to multiplicity k, while they pass simply through the $2k+1$ fundamental double points on d.

The image of any point on $\tau = 0$ is a definite point on the line $x_3 = 0$.

There are $2k+1$ conics of the system that pass through their associated points (z) on $x_3 = 0$; these are fundamental and the point (z) is transformed into itself, corresponding to the tangent to the associated conic passing through it. Hence the curve $\sigma = 0$ passes through each of these points. There are also two conics of the pencil which are tangent to $x_3 = 0$; their points of contact belong to $\sigma = 0$. These two categories account for all the intersections of $\sigma = 0$ and $x_3 = 0$.

The line $x_3 = 0$ is transformed into itself but not point for point.

The curve $\sigma = 0$ touches each conic of $\tau = 0$ at one of the intersections of $\sigma = 0$ and $x_3 = 0$.

The complete configuration of fundamental points consists of the four base points, each to multiplicity $k+1$, and the $2k+1$ double points on $x_3 = 0$. The image of each of the latter is the conic of the pencil $A\,B\,C\,D$ passing through it.

The image of a base point, as A, is a curve of order $2k+1$, having A

* **Ruffini, "Sulla risoluzione delle due equazioni di condizione delle trasformazioni Cremoniane piane delle figure piane," *Atti dell'Accademia di Bologna, Memorie* (3), Vol. 8 (1877), pp. 456-525.**

to multiplicity $k+1$ and passing k times through the other three base points and doubly through the $2k+1$ fundamental points on $x_3=0$. As (z) describes $x_3=0$, k conics of the pencil are uniquely determined by the tangent t at A; t and the associated line $A(z)$ are in $(1,k)$ correspondence and hence have $k+1$ self-corresponding elements in two concentric pencils.

The table of characteristics of this plane transformation may be expressed in the form

$$\begin{aligned}
&C_1 \sim C_{4k+3} : A^{2k+1}\, B^{2k+1}\, C^{2k+1}\, D^{2k+1}\, P_1{}^2 P_2{}^2 \cdots P^2{}_{2k+1}\\
&A \sim \alpha_{2k+1} : A^{k+1}\, B^k\, C^k\, D^k\, P_1 P_2 \cdots P_{2k+1}\\
&P_i \sim \pi_{2,i} : A\, B\, C\, D\, P_i\\
&J_{12k+6}(C_{4k+3}) : \alpha_{2k+1}\, \beta_{2k+1}\, \gamma_{2k+1}\, \delta_{2k+1}(2k+1)(\pi_{2,i})^2\\
&\tau_{2(2k+1)} : (2k+1)\pi_{2,i}\\
&\sigma_{2k+3} : A^{k+1}\, B^{k+1}\, C^{k+1}\, D^{k+1}\, P_1 P_2 \cdots P_{2k+1} (\text{genus } k+1).
\end{aligned}$$

4. *Image of points of* $\gamma_{4(n-1)}$. The image of a point on $\gamma_{4(n-1)}$ is the plane curve of order $2k+1$ just obtained in the $(1,k)$ correspondence of the points of d and a pencil of conics.

A plane meets its own image surface in a composite curve of order $2n(k+1)-1$, consisting of a curve of order $n(k+2)-1$, the intersection of $\sigma=0$ and the given plane; and k curves of order n, the intersections of the plane with the k surfaces of order n belonging to the point (z) of intersection of the given plane and d. Each of these curves of order n goes into itself by central projection, center on d.

5. *Conjugate of a line.* The image of an arbitrary straight line l is a curve $C_{2n(k+1)-1}$ of order $2n(k+1)-1$ which meets d in $2[n(k+1)-1]$ points, the images of the intersections of l and $\tau_{2[n(k+1)-1]}=0$. This $C_{2n(k+1)-1}$ also meets l in $n(k+2)-1$ points, the intersections of l and $\sigma_{n(k+2)-1}=0$. If l meets d, the image of l loses a curve of order $2(n-2)\times(k+1)$, the image of the point of intersection of d and l, as d is $2(n-2)(k+1)$ fold in the transformation. The proper conjugate is a curve of order $4k+3$, lying in the plane determined by d and l; it has four $(2k+1)$ fold points at the intersections of the plane and $\gamma_{4(n-1)}$ and $2k+1$ double points on d.

Every plane through d cuts $\tau_{2[n(k+1)-1]}=0$ in d counted $2(n-2)(k+1)$ times and in $2k+1$ conics, the images of the $2k+1$ fundamental points on d in that plane.

6. *Image of* $\gamma_{4(n-1)}$. The image of a surface of order n of the pencil is of order $n[2n(k+1)-1]$ and is composed of the given surface, of $\tau_{2[n(k+1)-1]}$ to multiplicity $(n-2)$, and of a surface $\Gamma_{4[n(k+1)-1]}$ of order

$4[n(k+1)-1]$, the image of $\gamma_{4(n-1)}$. $\Gamma_{4[n(k+1)-1]}$ contains $\gamma_{4(n-1)}$ to multiplicity $4k+1$ and d itself to multiplicity $4(n-2)(k+1)$. This can be seen from the fact that a plane through d cuts $\Gamma_{4[n(k+1)-1]}$ in a curve of order $4[n(k+1)-1]$; d intersects $\gamma_{4(n-1)}$ in $4(n-2)$ points; each point of $\gamma_{4(n-1)}$ goes into a plane curve of order $2k+1$; therefore d appears on $\Gamma_{4[n(k+1)-1]}$ to multiplicity $4(n-2)(k+1)$. $[4(nk+n-1)-4(2k+1) = 4(n-2)(k+1)]$.

7. *Points on d.* The line d is $2(n-2)(k+1)$ fold in the transformation, hence any point on d has an image curve of order $2(n-2)(k+1)$. But any point of d has an image conic in each of the $k(n-2)$ tangent planes of the k surfaces belonging to that point, that is, a curve of order $2k(n-2)$ is accounted for; the residual is d itself taken $2(n-2)$ times. Hence d is a fundamental line of the first kind and also of the second for $n > 2$. The equations of transformation show that any point on d goes into the whole line d.

When $n=2$, the line d is not a fundamental line. The resulting transformation in this case is described by Montesano.* In this case consider the base curve γ_4 of a pencil of quadrics and g a bisecant of γ_4. A point (y) on g determines the quadric H_g of the pencil containing g, and also determines the associated point (z) on d. The plane determined by (z) and g meets $H_g=0$ in a residual line g' through the two residual points of γ_4 in the plane. Thus, each quadric of the pencil is transformed into itself, and the two systems of reguli are interchanged. The $2k+1$ pairs of parasitic lines meet at $P_1, \cdots P_{2k+1}$ on d, hence these lines of each pair are also of different reguli.

8. *τ and σ tangent.* The surfaces $\tau_{2[n(k+1)-1]}=0$ and $\sigma_{n(k+2)-1}=0$ may be shown to be tangent along d, so that d counts for $2k$ units more in the common intersection. A plane passed through d intersects $\sigma_{n(k+2)-1}=0$ in a curve $C_{n(k+2)-1}$ of order $n(k+2)-1$, in which d is counted $(n-2)(k+2)$ times, leaving a curve C_{2k+3} of order $2k+3$. In the plane C_{2k+3} has $k+1$-fold points at each of A, B, C, and D, the intersections of the two conics cut from the base surfaces by the plane; and simple points in $P_1, P_2, \cdots, P_{2k+1}$, the $2k+1$ fundamental points on d in that plane. The plane intersects $\tau_{2[n(k+1)-1]}=0$ in $2k+1$ conics of the same pencil $A\,B\,C\,D$, which pass through $P_1, P_2 \cdots P_{2k+1}$, respectively. Each of these conics meets C_{2k+3} in $2(2k+3)$ points, consisting of $(k+1)$ each at A, B, C, and D and two

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* Montesano, "Su una classe di trasformazioni razionali ed involutorie dello spazio di genere arbitrario n e di grado $2n+1$," *Giornale Matematiche di Battaglini,* Vol. 31 (1893), pp. 36-50.

at a fundamental point P_i. Every point of the conic goes into P_i and no point of the conic lies on C_{2k+3} except the fundamental points; hence the section of $\sigma_{n(k+2)-1}=0$ must touch that conic at P_i; the surfaces $\tau_{2[n(k+1)-1]}=0$ and $\sigma_{n(k+2)-1}=0$ are tangent along d. In fact, d counts for $2k$ units more in the common intersection.

At any point (z) on d $2k(n-2)$ tangent planes to a surface of the pencil coincide in pairs and with the $k(n-2)$ tangent planes to the k surfaces of order n which are associated with that point. These $k(n-2)$ planes are also tangent to $\tau_{2[n(k+1)-1]}=0$ and to $\sigma_{n(k+2)-1}=0$, each simply. The fundamental conics in these planes lie on all the surfaces of the web $|S_{2n(k+1)-1}|$, conjugate to the field of planes of space.

The other planes associated with the same point and tangent to $S_{2n(k+1)-1}$ form a group of an involution of order $2(n-2)$

$$z_3[2(n-2)\text{ order in } x_1, x_2] + z_4[2(n-2)\text{ order in } x_1, x_2] = 0;$$

when (z) describes d, the group of planes describe the involution.

9. *Analytic method.* The details of this analysis will be given for $n=3$, $k=1$, since this case can be seen more readily than the general case.

The tangent plane to the cubic surface

$$z_4(x_1u+x_2v)-z_3(x_1w+x_2t)=0,$$

in which u, v, w, and t are quadratic forms in (x_1, x_2, x_3, x_4), at the point $(0, 0, z_3, z_4)$ is

$$z_4[x_1u(z)+x_2v(z)]-z_3[x_1w(z)+x_2t(z)]=0.$$

A plane $(ax)=0 \sim \tau(ax)+\sigma[a_3(x_1u+x_2v)+a_4(x_1w+x_2t)]=0$. Call this image surface S_{11}. The configuration of the tangent planes to $\sigma_8=0$, $\tau_{10}=0$, and $S_{11}=0$ is found from those terms in their equations which contain the highest powers of x_3, x_4. These appear to degrees five, six, and seven, respectively, in σ_8, τ_{10}, and S_{11}. When z_3 and z_4 are substituted for x_3 and x_4, respectively, in these terms of σ_8, τ_{10}, and S_{11}, the results denoted by σ_0, τ_0, and S_0, respectively, have the forms

$$\begin{aligned}
\sigma_0 &\equiv -2[z_4(x_1u(z)+x_2v(z))-z_3(x_1w(z)+x_2t(z))] \\
&\quad\times[Az_3^2+Bz_3z_4+Cz_4^2]=0, \\
\tau_0 &\equiv [z_4(x_1u(z)+x_2v(z))-z_3(x_1w(z)+x_2t(z))] \\
&\quad\times[2Az_3(x_1u(z)+x_2v(z)) + B\{z_4(x_1u(z)+x_2v(z)) \\
&\qquad + z_3(x_1w(z)+x_2t(z))\} + 2Cz_4(x_1w(z)+x_2t(z))]=0, \\
S_0 &\equiv [z_4(x_1u(z)+x_2v(z))-z_3(x_1w(z)+x_2t(z))]^2 \\
&\quad\times[a_4(2Az_3+Bz_4)-a_3(Bz_3+2Cz_4)]=0,
\end{aligned}$$

where

$$A \equiv (x_1 w_{34} + x_2 t_{34})(x_1 u_{33} + x_2 v_{33}) - (x_1 u_{34} + x_2 v_{34})(x_1 w_{33} + x_2 t_{33})$$
$$B \equiv (x_1 w_{44} + x_2 t_{44})(x_1 u_{33} + x_2 v_{33}) - (x_1 u_{44} + x_2 v_{44})(x_1 w_{33} + x_2 t_{33})$$
$$C \equiv (x_1 w_{44} + x_2 t_{44})(x_1 u_{34} + x_2 v_{34}) - (x_1 u_{44} + x_2 v_{44})(x_1 w_{34} + x_2 t_{34}).$$

Hence from the above forms it is seen that of the four tangent planes to $S_{11} = 0$ at any point $(0, 0, z_3, z_4)$ on d, two coincide with each other and with the tangent plane to the cubic surface $z_4(x_1 u + x_2 v) - z_3(x_1 w + x_2 t) = 0$ associated with that point. This plane is also tangent to $\tau_{10} = 0$ and $\sigma_8 = 0$, each simply. The fundamental conic in this plane lies on all the $|S_{11}|$ of the ∞^2 system, conjugate to the bundle of planes passing through the associated point on d.

The equations of the other two planes associated with the same point and tangent to $S_{11} = 0$ contain the coördinates of (z) simply; $a_3(Bz_3 + 2Cz_4) - a_4(2Az_3 + Bz_4) = 0$; hence when (z) describes d, the pair of planes describes an involution of order two.

Thus in the general case, the image of any point of d is a curve of order $2(n-2)(k+1)$, consisting of $k(n-2)$ conics and of the line d taken $2(n-2)$ times.*

Any two $S_{2n(k+1)-1}$ of the web $|S_{2n(k+1)-1}|$ intersect in d to multiplicity $4(n-2)^2[(k+1)^2+1]$. Every surface of the system has $(n-2)$ consecutive double lines d; the $k(n-2)$ tangent planes each counted twice being those of the associated surface of order n and being such double tangent planes for every $S_{2n(k+1)-1}$ of the system.

10. *Parasitic lines.* There are $[(6n-8)(k+1)-2]$ parasitic lines or fundamental straight lines of the second kind occurring simply on $\sigma_{n(k+2)-1} = 0$, $\tau_{2[n(k+1)-1]} = 0$, and hence on every $S_{2n(k+1)-1}$ of the web; they appear doubly on $\Gamma_{4[n(k+1)-1]} = 0$. The proof for determining the number of these parasitic lines may be given as follows:

The equation of the $n-2$ tangent planes to a surface

$$\lambda_2 \sum_{\mu=0}^{n-2} x_1^{n-\mu-2} x_2^{\mu} u_\mu(x) - \lambda_1 \sum_{\mu=0}^{n-2} x_1^{n-\mu-2} x_2^{\mu} v_\mu(x) = 0$$

of the pencil (1) at the point $(0, 0, z_3, z_4) \equiv (0, 0, \phi_2(\lambda), \phi_1(\lambda))$ is

$$\lambda_2 \sum_{\mu=0}^{n-2} x_1^{n-\mu-2} x_2^{\mu} u_\mu(\phi) - \lambda_1 \sum_{\mu=0}^{n-2} x_1^{n-\mu-2} x_2^{\mu} v_\mu(\phi) = 0. \tag{10}$$

Let one of these planes be expressed in the form $x_2 = rx_1$. When this sub-

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* Montesano, "Sulla teoria generale delle corrispondenze birazionali dello spazio," *Rendiconti della R. Accademia dei Lincei*, Vol. 5 (1918), pp. 396-400, 438-441.

stitution is made in the equation of the surface (1), and the necessary reduction is performed, the result is

$$\lambda_2 \sum_{\mu=0}^{n-2} r^\mu u_\mu(x) - \lambda_1 \sum_{\mu=0}^{n-2} r^\mu v_\mu(x) = 0. \tag{11}$$

Let (11) be represented by $C_2 \equiv \sum_{i,k=1}^{3} a_{ik}x_ix_k = 0$. It is seen that a_{11} is a coefficient of order n in r and one in λ; a_{13} and a_{14} of order $n-1$ in r and one in λ; and a_{33}, a_{44}, and a_{34} of order $n-2$ in r and one in λ. The discriminant has highest powers of r, λ, thus:

$$\Delta \equiv \begin{vmatrix} r^n, \lambda & r^{n-1}, \lambda & r^{n-1}, \lambda \\ r^{n-1}, \lambda & r^{n-2}, \lambda & r^{n-2}, \lambda \\ r^{n-1}, \lambda & r^{n-2}, \lambda & r^{n-2}, \lambda \end{vmatrix},$$

hence it contains r to power $3n-4$ and λ to power three.

The tangent planes (10) contain r to power $n-2$ and λ to power $2k+1$, since $\phi(\lambda)$ is of order $2k$ in λ.

When λ is eliminated between the equation of the tangent planes (10)

$$\lambda_2 \sum_{\mu=0}^{n-2} r^\mu u(\phi) - \lambda_1 \sum_{\mu=0}^{n-2} r^\mu v_\mu(\phi) \equiv A_0 \quad \lambda^{2k+1} + A_1\lambda^{2k} + \cdots + A_{2k+1} = 0$$

and

$$\Delta \equiv B_0\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0,$$

the resultant is of the form $\Sigma A_i^3 \cdot B_j^{2k+1}$, where A_i is of order $n-2$ in r and B_j is of order $3n-4$ in r. Therefore there are

$$3(n-2) + (2k+1)(3n-4) = [(6n-8)(k+1)-2]$$

parasitic lines.

The plane of any one of these lines g and d is tangent to its associated surface $F_n(x) = 0$ at $(g, d) = (z)$. The residual conic cut from the surface by this plane is composite, one component g passing through (z). Any point of g goes into the whole line g; the other component is the image of the point (z).

11. *Table of characteristics.* The general table of characteristics has the form

$$S_1 \sim S_{2n(k+1)-1} \; : \; d^{2(n-2)(k+1)}k(n-2)\bar{d}^2\gamma^{2k+1}_{4(n-1)}[(6n-8)(k+1)-2]g,$$

$$d \sim \tau_{2[n(k+1)-1]} : \; d^{2(n-2)(k+1)}k(n-2)\bar{d}\gamma^{2k+1}_{4(n-1)}[(6n-8)(k+1)-2]g,$$

$$\sigma_{n(k+2)-1} \sim \sigma_{n(k+2)-1} \; : \; d^{(n-2)(k+2)}k(n-2)\bar{d}\gamma^{k+1}_{4(n-1)}[(6n-8)(k+1)-2]g,$$

$$\gamma_{4(n-1)} \sim \Gamma_{4[n(k+1)-1]} \; : \; d^{4(n-2)(k+1)}k(n-2)\bar{d}^4\gamma^{4k+1}_{4(n-1)}[(6n-8)(k+1)-2]g^2,$$

$$J_{8[n(k+1)-1]} : \qquad \Gamma_{4[n(k+1)-1]} \qquad \tau^2_{2[n(k+1)-1]}.$$

12. *Intersections of principal surfaces.* A detailed account of the intersections of the various principal elements with each other in the transformation may now be explained.

Two $S_{2n(k+1)-1}$ of the web intersect in a curve of order $2n(k+1)-1$, the image of a straight line which is the intersection of the two planes, the conjugates of the two given surfaces; d to multiplicity $[2(n-2)(k+1)]^2$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)^2$; the $(6n-8)(k+1)-2$ parasitic lines; and a consecutive double line in each of the k sheets of the two surfaces.

An $S_{2n(k+1)-1}$ of the web and $\tau_{2[n(k+1)-1]}$ intersect in d to multiplicity $4(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)^2$; the $(6n-8)\times(k+1)-2$ parasitic lines; and $k(n-2)$ conics and d counted $2k(n-2)$ times, the images of a point on d which is the intersection of a plane and d, the conjugates of the given surfaces and $\tau_{2[n(k+1)-1]}$, respectively.

An $S_{2n(k+1)-1}$ of the web and $\sigma_{n(k+2)-1}$ intersect in d to multiplicity $2(n-2)^2(k+1)(k+2)$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(k+1)$; $(6n-8)(k+1)-2$ parasitic lines; a curve of order $n(k+2)-1$, the intersection of a plane and $\sigma_{n(k+2)-1}$, the conjugates of the given surfaces; and d which is double on the k sheets of $S_{2n(k+1)-1}$.

An $S_{2n(k+1)-1}$ of the web and $\Gamma_{4[n(k+1)-1]}$ intersect in d to multiplicity $4(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(4k+1)$; the $(6n-8)\times(k+1)-2$ parasitic lines twice; $4(n-1)$ curves of order $2k+1$, the images of the points of intersection of a plane and $\gamma_{4(n-1)}$, which are conjugates of the given surfaces; and d taken $8k(n-2)$ times, as d is a consecutive double line on $S_{2n(k+1)-1}$ and a consecutive four-fold line on $\Gamma_{4[n(k+1)-1]}$.

$\tau_{2[n(k+1)-1]}$ and $\sigma_{n(k+2)-1}$ intersect in d to multiplicity $2(n-2)^2(k+1)\times(k+2)$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(k+1)$; the $[(6n-8)\times(k+1)-2]$ parasitic lines; and d taken $2k(n-2)$ times, since d counts for $2k$ units more in the common intersection.

$\tau_{2[n(k+1)-1]}$ and $\Gamma_{4[n(k+1)-1]}$ intersect in d to multiplicity $8(n-2)^2\times(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(4k+1)$; the $(6n-8)\times(k+1)-2$ parasitic lines twice; $4k(n-2)$ conics in the four tangent planes of the k surfaces associated with the points in which $\gamma_{4(n-1)}$ meets d; and d taken $4k(n-2)$ times, since d is four-fold on k sheets of $\Gamma_{4[n(k+1)-1]}$
$\times(k+1)$; $\gamma_{4(n-1)}$ to multiplicity $(k+1)(4k+1)$; $(6n-8)(k+1)-2$

$\sigma_{n(k+2)-1}$ and $\Gamma_{4[n(k+1)-1]}$ intersect in d to multiplicity $4(n-2)^2(k+2)$ and single on $\tau_{2[n(k+1)-1]}$.
parasitic lines twice; d taken $4k(n-2)$ times, since d counts as four-fold on k sheets of $\Gamma_{4[n(k+1)-1]}$; and $\gamma_{4(n-1)}$ taken $k+1$ times, accounting for the intersection of $\gamma_{4(n-1)}$ and $\sigma_{n(k+2)-1}$, the conjugates of $\Gamma_{4[n(k+1)-1]}$ and $\sigma_{n(k+2)-1}$, respectively.

13. *Transforms of principal elements.* The transforms of the principal elements may be accounted for analytically. Since the transformation is involutorial, an $S_{2n(k+1)-1}$ of the web must go into a plane; d into $\tau_{2[n(k+1)-1]}$; $\gamma_{4(n-1)}$ into $\Gamma_{4[n(k+1)-1]}$; and since d is a consecutive double line on the given surface, the image of d, which is $\tau_{2[n(k+1)-1]}$, appears twice.

$$S_{2n(k+1)-1} : d^{2(n-2)(k+1)} k(n-2) \bar{d}^2 \gamma_{4(n-1)}^{2k+1} [(6n-8)(k+1)-2] g.$$

The image of $\sigma_{n(k+2)-1}$ under I is itself; the image of $\tau_{2[n(k+1)-1]}$ is d; and the image of $\Gamma_{4[n(k+1)-1]}$ is $\gamma_{4(n-1)}$.

Thus, the intersections of the principal surfaces and the transforms of the principal elements confirm all the details of the Table of Characteristics.

CORNELL UNIVERSITY.

MINIMAL SURFACES OF UNIPLANAR DERIVATION.

By E. F. Beckenbach.*

If the coördinates of a surface

$$(1) \qquad x_j = x_j(u, v), \qquad (j = 1, 2, 3),$$

defined in the circle

$$(2) \qquad R = [(u - u_0)^2 + (v - v_0)^2]^{1/2} < \rho,$$

are expressed in terms of isothermic parameters, that is, if

$$(3) \qquad E = G, \quad F = 0,$$

then a necessary and sufficient condition that the surface be minimal is that the functions (1) be harmonic.

This being so, these functions are the real parts of analytic functions,

$$(4) \qquad x_j = \Re f_j(z), \qquad (z = u + iv).$$

Now (3) becomes

$$(5) \qquad \sum_{j=1}^{3} f'^2_j = 0.$$

If the surface lies on the plane

$$x_3 = 0,$$

equation (5) becomes equivalent to the Cauchy-Riemann differential equations involving x_1 and x_2. Consequently, a map of (2) given by an analytic function of a complex variable is a special case of an isothermic map of (2) on a minimal surface. Since, further, minimal surfaces possess many of the properties of maps given by analytic functions, minimal surfaces well might be said to constitute the space analogue of maps given by analytic functions.

The purpose of this paper is to define, and to point out some of the properties of, a class of minimal surfaces which seems to correspond closely to that class of maps of (2), given by analytic functions of the complex variable z, which lie on a single-sheeted plane, or, as we shall say, on uniplanar † regions. An equivalent characterization of these functions is that they take on no value more than once in (2). For a discussion of such functions, see, among others, L. Bieberbach, *Lehrbuch der Funktionentheorie*,

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Vol. II, pp. 82-94 (1927), or L. R. Ford, *Automorphic Functions*, pp. 169-177 (1929).

If and only if it is possible so to choose the coördinate axes that the functions

$$M(z) = f_1 + if_2, \quad N(z) = f_1 - if_2 \tag{6}$$

both map (2) *on uniplanar regions, we say that the functions* (1) *map* (2) *on a minimal surface of uniplanar derivation.*

This is a large but by no means exhaustive class of minimal surfaces. For example, the functions

$$x_1 = \Re\left[z + \tfrac{1}{4}z^2\right],$$

$$x_2 = \Re\left[\frac{i}{4}z^2\right],$$

$$x_3 = \Re\left[\frac{2i}{3}(1+z)^{3/2}\right],$$

map the unit circle $|z| < 1$ on a minimal surface of uniplanar derivation, so that, as we shall show, this surface possesses many of the properties of uniplanar analytic maps; but the surface does not lie on a plane.

If the functions (6) give uniplanar maps with one choice of coördinates, it does not follow that necessarily they give uniplanar maps with all choices. Thus, for the functions

$$f_1 = z, \quad f_2 = 0, \quad f_3 = -iz$$

we have

$$M(z) = z \text{ and } N(z) = z$$

both mapping (2) on uniplanar regions; but

$$f_1 - if_3 = 0.$$

We shall assume in the future that unless otherwise stated the coördinates of a minimal surface of uniplanar derivation have been so chosen that $M(z)$ and $N(z)$ both give uniplanar maps of the region of definition.

The following theorem will help justify our definition.

A uniplanar analytic map of (2) *is a minimal surface of uniplanar derivation. Conversely, if a minimal surface of uniplanar derivation lies on a Riemann surface, it lies on a uniplanar region.*

Suppose first that we have a uniplanar analytic map of (2). As we have pointed out, this is a special case of a minimal surface. If we choose coördinates so that the map lies in the (x_1, x_3)-plane, then

$$f_1 = x_1 + ix_3,$$
$$f_2 = 0,$$
$$f_3 = x_3 - ix_1 = - if_1,$$

and the functions (6) both give the uniplanar map in question. Consequently, the map is a minimal surface of uniplanar derivation.

Suppose secondly that we have (2) mapped conformally on a minimal surface lying on a Riemann surface but not on a single-sheeted plane. We shall show that the surface is not of uniplanar derivation. With the same axes as in the preceding case, the map is given by the analytic function

$$f_1 = x_1 + ix_3.$$

We must show that for any transformation of the rectangular axes,

$$X_j = \sum_{k=1}^{3} a_{j,k} x_k, \qquad (j = 1, 2, 3),$$

at least one of the new functions (6) does not give a uniplanar map.

The surface now is given by

$$X_j = a_{j,1} x_1(u, v) + a_{j,3} x_3(u, v) = \Re F_j(z).$$

Here

$$F_j = a_{j,1} f_1 + a_{j,3} f_3 = (a_{j,1} - ia_{j,3}) f_1,$$

so that

$$F_1 + iF_2 = (a_{1,1} - ia_{1,3} + ia_{2,1} + a_{2,3}) f_1,$$
$$F_1 - iF_2 = (a_{1,1} - ia_{1,3} - ia_{2,1} - a_{2,3}) f_1.$$

Neither of these functions gives a uniplanar map, since f_1 does not give such a map. The surface, therefore, is not of uniplanar derivation.

It is an interesting but obvious observation that

a) *Every member of the family of real minimal surfaces associate to a minimal surface of uniplanar derivation is of uniplanar derivation.*

b) *Every real surface homothetic to a minimal surface of uniplanar derivation is a minimal surface of uniplanar derivation.*

If the original surface is given by (4), then the associate real minimal surfaces are given by

$$y_j = \Re e^{it} f_j(z),$$

where the parameter t is real. This family consists, to within their positions in space, of all the real minimal surfaces applicable to the first surface.

The homothetic real surfaces are given, to within their positions in space, by

$$y_j = \mathfrak{R} c f_j(z),$$

where the parameter c is real and not zero.

Combining the two families, we have that

$$y_j = \mathfrak{R} c e^{it} f_j(z) = \mathfrak{R} w f_j(z), \tag{7}$$

where w is a non-vanishing complex parameter, define the two parameter family of minimal surfaces of uniplanar derivation applicable, or applicable after a magnification, to the minimal surface of uniplanar derivation defined by (4).

Obviously these surfaces (7) are minimal, for the y_j are the real parts of analytic functions and so are harmonic and, by (5),

$$w^2 \sum_{j=1}^{3} f'^2_j = 0.$$

Finally, the condition that they be of uniplanar derivation is that the functions

$$w(f_1 + if_2), \qquad w(f_1 - if_2)$$

give uniplanar maps, a condition that is satisfied since both of (6) give uniplanar maps, and since $w \neq 0$.

Every simply connected subregion of a minimal surface of uniplanar derivation is of uniplanar derivation.

Let the isothermic harmonic functions (1) map the circle (2) on the minimal surface of uniplanar derivation S, and consider any simply connected subregion S_1 of S. In this mapping, a subregion S'_1 of (2) is mapped on S_1. Let the function

$$z = z(Z), \qquad Z = U + iV,$$

map the circle

$$[(U - U_0)^2 + (V - V_0)^2]^{1/2} < \rho \tag{2'}$$

on S'_1. Since the functions (6) give uniplanar maps of (2), they give uniplanar maps of any part of (2), in particular of S'_1. Consequently, the functions

$$x_j = \mathfrak{R} f_j(z) = \mathfrak{R} f_j\,[z(Z)] = \mathfrak{R} F_j(Z),$$

which map (2′) on S_1, are such that

$$F_1 + iF_2 \quad \text{and} \quad F_1 - iF_2$$

give uniplanar maps of (2′).

If an analytic function, not identically constant, has a zero derivative

at any point, this point is a branch point of the function and the map is not uniplanar. Only at such points does the conformal character of the mapping break down. Similarly,

If the isothermic harmonic functions (1) *map the circle* (2) *on a minimal surface of uniplanar derivation, the mapping is conformal throughout.*

The mapping is conformal except where

$$E = 0,$$

that is, where simultaneously

$$f'_1 = 0, \quad f'_2 = 0, \quad f'_3 = 0,$$

since

$$E = \tfrac{1}{2} \sum_{j=1}^{3} | f'_j |^2.$$

The points of (2) where this condition is satisfied necessarily are isolated for any minimal surface. But if there are such points, the functions (6) do not give uniplanar maps and consequently the surface is not of uniplanar derivation.

The existence of such a singular point is a sufficient, but not a necessary, condition that a map be not uniplanar or of uniplanar derivation.

The equations of Weierstrass for a general minimal surface,

$$x_1 = \Re \int i\,[g^2(z) - h^2(z)]\, dz,$$

$$x_2 = \Re \int [g^2(z) + h^2(z)]\, dz,$$

$$x_3 = \Re \int 2ig(z)h(z)dz,$$

are obtained from (5) by means of the equations of definition

$$f'_1 + if'_2 = 2ig^2(z),$$
$$f'_1 - if'_2 = -2ih^2(z).$$

Hence, for minimal surfaces of uniplanar derivation,

$$\text{(8)} \qquad \begin{aligned} \phi(z) &= \int g^2(z)dz = (1/2i)M(z), \\ \psi(z) &= \int h^2(z)dz = (-1/2i)N(z), \end{aligned}$$

both give uniplanar maps.

A direct computation gives

$$E = [g(z)\bar{g}(z) + h(z)\bar{h}(z)]^2,$$

where $\bar{\theta}$ denotes the conjugate imaginary of θ.

The length of a curve on the minimal surface is given by

$$L = \int_{z_0}^{z_1} E^{1/2} \mid dz \mid = \int_{z_0}^{z_1} \mid g \mid^2 \mid dz \mid + \int_{z_0}^{z_1} \mid h \mid^2 \mid dz \mid .$$

The lengths of curves on the maps

$$w_1 = \phi(z) \quad \text{and} \quad w_2 = \psi(z)$$

are given by

$$L_1 = \int_{z_0}^{z_1} \mid g \mid^2 \mid dz \mid \quad \text{and} \quad L_2 = \int_{z_0}^{z_1} \mid h \mid^2 \mid dz \mid$$

respectively. Consequently, if the paths in (2) are the same in the three cases,

$$L = L_1 + L_2.$$

The following theorem is analogous to Bieberbach's famous theorem on uniplanar maps.*

If the isothermic harmonic functions (1) *map the circle* (2) *on a finite surface of uniplanar derivation, then the minimum distance on the surface from the image of* (u_0, v_0) *to the boundary is greater than or at least equal to* $E_0^{1/2}\rho/4$, *where* E_0 *designates the value of* E *at* (u_0, v_0). *No closer inequality holds for all surfaces of this type.*

The above distance is given by the minimum, for all paths of integration, of the integral

$$\int_{R=0}^{\rho} E^{1/2} \mid dz \mid = \int_{R=0}^{\rho} \mid g \mid^2 \mid dz \mid + \int_{R=0}^{\rho} \mid h \mid^2 \mid dz \mid .$$

Since the functions $\phi(z)$ and $\psi(z)$ give finite uniplanar maps of (2), we can apply Bieberbach's theorem to these functions, getting, for any path of integration,

$$(9) \qquad \begin{aligned} &\int_{R=0}^{\rho} \mid g \mid^2 \mid dz \mid \geq \mid g_0 \mid^2 \rho/4, \\ &\int_{R=0}^{\rho} \mid h \mid^2 \mid dz \mid \geq \mid h_0 \mid^2 \rho/4, \end{aligned}$$

where g_0 and h_0 are the values of g and h at (u_0, v_0).

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* Bieberbach, *loc. cit.*, p. 86; Ford, *loc. cit.*, p. 169.

Adding the two inequalities (9) and applying

$$E_0^{1/2} = |g_0|^2 + |h_0|^2,$$

we obtain the desired inequality

$$\int_{R=0}^{\rho} E^{1/2}\,|dz| \geq E_0^{1/2}\,\rho/4. \tag{10}$$

Since, by Bieberbach's theorem, the inequalities (9) are the closest possible for all finite uniplanar analytic maps, and since uniplanar maps are a special case of maps of uniplanar derivation, it follows that (10) is the closest inequality possible for all finite minimal surfaces of uniplanar derivation.

The equality in the preceding theorem,

$$\textit{minimum} \int_{R=0}^{\rho} E^{1/2}\,|dz| = E_0^{1/2}\,\rho/4, \tag{11}$$

can hold only if the map of uniplanar derivation is actually uniplanar.

For (11) to hold, each equality in (9) must hold for a certain path of integration; furthermore, this path must be the same in both cases. Consequently, $\phi(z)$ and $\psi(z)$ are derived from Koebe's function

$$W = Z/(1 + e^{-i\alpha}Z)^2, \qquad z = z_0 + \rho Z,$$

by means of

$$W = [\phi(z_0 + \rho Z) - \phi(z_0)]/\rho g_0^2,$$

and

$$W = [\psi(z_0 + \rho Z) - \psi(z_0)]/\rho h_0^2,$$

respectively, where the real parameter α has the same value in both cases and where

$$z_0 = u_0 + iv_0.$$

It is no restriction on the generality to take

$$\phi(z_0) = 0, \qquad \psi(z_0) = 0,$$

so that

$$\psi(z) = (h_0^2/g_0^2)\phi(z). \tag{12}$$

From (6), (8), and (12),

$$f_1 + if_2 = 2i\phi,$$
$$f_1 - if_2 = -2i(h_0^2/g_0^2)\phi,$$

so that, by (5),

$$
\begin{aligned}
&f_1 = i(1 - h_0^2/g_0^2)\phi,\\
(13)\qquad &f_2 = (1 + h_0^2/g_0^2)\phi,\\
&f_3 = \pm 2i(h_0/g_0)\phi.
\end{aligned}
$$

Setting

$$
\begin{aligned}
\phi &= A(u, v) + iB(u, v) = A + iB,\\
h_0/g_0 &= a + ib,
\end{aligned}
$$

and using (4), we find from (13) that

$$
\begin{aligned}
x_1 &= 2abA - (1 - a^2 + b^2)B,\\
x_2 &= (1 + a^2 - b^2)A - 2abB,\\
x_3 &= \mp 2bA \mp 2aB.
\end{aligned}
$$

The transformation of the rectangular coördinates defined by

$$
\begin{aligned}
X_1 &= \frac{b}{(a^2+b^2)^{1/2}} x_1 + \frac{a}{(a^2+b^2)^{1/2}} x_2,\\
X_2 &= \frac{-a(1-a^2-b^2)}{(a^2+b^2)^{1/2}(1+a^2+b^2)} x_1 + \frac{b(1-a^2-b^2)}{(a^2+b^2)^{1/2}(1+a^2+b^2)} x_2\\
&\qquad \mp \frac{2(a^2+b^2)}{(a^2+b^2)^{1/2}(1+a^2+b^2)} x_3,\\
X_3 &= \frac{2a}{1+a^2+b^2} x_1 - \frac{2b}{1+a^2+b^2} x_2 \pm \frac{a^2+b^2-1}{1+a^2+b^2} x_3,
\end{aligned}
$$

preserves orthogonality of axes and yields

$$
\begin{aligned}
X_1(u, v) &= (aA - bB)\,\frac{1+a^2+b^2}{(a^2+b^2)^{1/2}},\\
X_2(u, v) &= (bA + aB)\,\frac{1+a^2+b^2}{(a^2+b^2)^{1/2}},\\
X_3(u, v) &= 0;
\end{aligned}
$$

whence,

$$
\begin{aligned}
(14)\qquad X_1 + iX_2 &= \frac{1+a^2+b^2}{(a^2+b^2)^{1/2}}(a+ib)[A(u, v) + iB(u, v)]\\
&= E_0^{1/2} e^{i(\theta_1+\theta_2)} \rho W,
\end{aligned}
$$

where

$$
g_0 = e^{i\theta_1}\,|\,g_0\,|, \qquad h_0 = e^{i\theta_2}\,|\,h_0\,|.
$$

The function (14) gives a uniplanar analytic map of (2), since W gives such a map.

We now establish the following deformation theorem.

If the isothermic harmonic functions (1) *map the circle* (2) *on a finite*

surface of uniplanar derivation, then at any point within the circle the following inequalities hold,

$$(15)\qquad \frac{1-r}{(1+r)^3} \leq \left[\frac{E(u,v)}{E_0}\right]^{\frac{1}{2}} \leq \frac{1+r}{(1-r)^3},$$

where

$$(16)\qquad [(u-u_0)^2 + (v-v_0)^2]^{\frac{1}{2}} = r\rho.$$

Further, no closer inequalities hold for all surfaces of this type.

We know that *

$$(17)\qquad \begin{aligned} \frac{1-r}{(1+r)^3} &\leq \frac{|g(u,v)|^2}{|g_0|^2} \leq \frac{1+r}{(1-r)^3}, \\ \frac{1-r}{(1+r)^3} &\leq \frac{|h(u,v)|^2}{|h_0|^2} \leq \frac{1+r}{(1-r)^3}. \end{aligned}$$

It is a simple algebraic fact that

$$\frac{|g|^2 + |h|^2}{|g_0|^2 + |h_0|^2}$$

lies between

$$|g|^2/|g_0|^2 \quad \text{and} \quad |h|^2/|h_0|^2,$$

whence (15).

The equalities in (17) and (15) hold only for those functions for which the equalities hold in the preceding theorem. The conclusions follow as before that no closer inequalities hold for all surfaces of this type and that if the equalities hold the map is uniplanar, given by (14).

If (u_1, v_1) and (u_2, v_2) lie at the same distance $r\rho$ from (u_0, v_0), we obtain from (15) the inequality

$$\left(\frac{1-r}{1+r}\right)^4 \leq \left[\frac{E(u_1,v_1)}{E(u_2,v_2)}\right]^{\frac{1}{2}} \leq \left(\frac{1+r}{1-r}\right)^4.$$

If the isothermic harmonic functions (1) *map the circle* (2) *on a finite surface of uniplanar derivation, then the minimum distance* $m(u,v)$ *on the surface from the image of* (u_0, v_0) *to the point corresponding to* (u,v) *satisfies the inequalities*

$$(18)\qquad [r/(1+r)^2]\,\rho E_0^{\frac{1}{2}} \leqq m(u,v) \leqq [r/(1-r)^2]\,\rho E_0^{\frac{1}{2}},$$

where r *is given by* (16). *No closer inequalities hold for all surfaces of this type.*

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* Bieberbach, *loc. cit.*, p. 88; Ford, *loc. cit.*, pp. 172-173.

The second inequality of (15) gives, for any path of integration,

$$\int_{R=0}^{r} E^{1/2}\,|\,dz\,| \leq \int_{R=0}^{r} E_0^{1/2}\,[(1+r)/(1-r)^3]\,|\,dz\,|\,.$$

The first of these integrals, taken along a radius, is at least as great as $m(u,v)$. Hence,

$$(19)\qquad m(u,v) \leq \int_0^r [(1+r)/(1-r)^3]\,\rho E_0^{1/2}\,dr = [r/(1-r)^2]\,\rho E_0^{1/2}.$$

If the curve C minimizes the distance, then

$$m(u,v) = \int_C E^{1/2}\,|\,dz\,|\,.$$

This integral is not increased if we substitute for

$$|\,dz\,| = [\rho^2 dr^2 + \rho^2 r^2 d\theta^2]^{1/2}$$

the not larger quantity $\rho\,|\,dr\,|$:

$$m(u,v) \geq \int_C \rho E^{1/2}\,|\,dr\,|\,.$$

The first inequality of (15) gives, then,

$$(20)\qquad \begin{aligned} m(u,v) &\geq \int_C [(1-r)/(1+r)^3]\,\rho E_0^{1/2}\,|\,dr\,| \\ &\geq \int_0^r [(1-r)/(1+r)^3]\,\rho E_0^{1/2}\,dr \\ &= [r/(1+r)^2]\,\rho E_0^{1/2}. \end{aligned}$$

From (19) and (20) we get (18). The equalities hold only as in the preceding two theorems.

These last two theorems yield the following pair of theorems. The proofs, depending on a division of the (u,v) region into squares of sufficiently small size, are exactly the same as those of the corresponding theorems concerning uniplanar maps, and therefore are not given here.*

Let Σ' be a uniplanar finite region and let Σ be a closed point set consisting only of interior points of Σ'. Let the isothermic harmonic functions (1) map Σ' on a finite surface of uniplanar derivation. Then there exists a constant K, dependent on Σ and Σ' but independent of the mapping functions, such that if (u_1, v_1) and (u_2, v_2) are any two points of Σ, then

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* Bieberbach, *loc. cit.*, p. 89; Ford, *loc. cit.*, pp. 175-177.

$$\frac{1}{K} < \frac{E(u_1, v_1)}{E(u_2, v_2)} < K.$$

Finally,

In the mapping of the preceding theorem there exists a constant L, independent of the mapping functions, such that if

$$(u_1, v_1), \quad (u_2, v_2), \quad \text{and} \quad (u_3, v_3)$$

are any three points of Σ, *then*

$$m\,[(u_1, v_1),\ (u_2, v_2)] < L\,[E(u_3, v_3)]^{1/2},$$

where

$$m\,[(u_1, v_1),\ (u_2, v_2)]$$

denotes the minimum distance on the surface between the points (u_1, v_1) *and* (u_2, v_2).

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CONCERNING REGULAR PSEUDO *D*-CYCLIC SETS.

By Leonard M. Blumenthal.

1. *Introduction.* A set of points is called pseudo d-cyclic provided that each three of the points is congruent with three points of a circle of metric diameter d, while the whole set is not congruent to a subset of the circle.* In a recent paper † the writer has completely characterized those pseudo d-cyclic sets that (1) contain no convex tripod, and (2) contain no pseudo d-cyclic quadruples that are pseudo-linear.‡ Such pseudo d-cyclic sets were called *proper.* The purpose of this paper is to characterize pseudo d-cyclic sets that contain no convex tripods, the assumption that the set contains no pseudo-linear quadruples that are pseudo d-cyclic not being made. We call such sets *regular.* The principal theorem of this paper proves that these sets are equilateral provided that they contain more than four points.

It has been shown in the paper referred to above that pseudo d-cyclic quadruples are of three kinds; namely, (1) pseudo d-cyclic quadruples that contain no linear triples, (2) pseudo d-cyclic quadruples that contain exactly three linear triples, and (3) pseudo d-cyclic quadruples that have all four of the triples they contain linear. The second case can occur only when the quadruple forms a convex tripod, and hence this case is excluded from this discussion. The third case occurs when the quadruple is pseudo-linear, and no two of its points are diametral. We shall refer to pseudo d-cyclic quadruples as being of the *first, second,* or *third kinds,* depending upon whether they are in the first, second, or third of the above classifications, respectively. Further, if p_1, p_2, p_3, p_4 is a pseudo d-cyclic quadruple of either the first or third kind, then $p_1p_2 = p_3p_4$, $p_1p_3 = p_2p_4$, $p_2p_3 = p_1p_4$.

* Throughout this paper the word "circle" refers to the circumference. Distance between two points of a circle is defined to be the length of the shorter arc of the circle joining them.

† "A Complete Characterization of Proper Pseudo *D*-Cyclic Sets of Points," *American Journal of Mathematics*, Vol. 54 (1932), pp. 387-396.

‡ Four points form a convex tripod provided that one of the points lies *between* each of the three pairs of points contained in the remaining three points. (A point q is said to lie between two points p, r provided that $pq + qr = pr$. The triple p, q, r is said to be *linear.* We shall denote the above relation by the notation pqr.) Four points form a pseudo-linear set provided that each three of the points is congruent to three points of a line, while the four points are not congruent to four points of a line.

2. In this section, we establish two lemmas and a theorem concerning pseudo d-cyclic quadruples and quintuples.

LEMMA 1. *A pseudo d-cyclic quadruple does not contain two diametral points.*

To prove this, we suppose that two points, say p_1, p_3, of the pseudo d-cyclic quadruple p_1, p_2, p_3, p_4 are diametral; that is, the distance p_1p_3 equals d. Then evidently the two triples p_1, p_2, p_3 and p_1, p_4, p_3 containing these two points are linear, and we have $p_1p_2 + p_2p_3 = p_1p_3$; $p_1p_4 + p_4p_3 = p_1p_3$. Now the quadruple is either of the second or the third kind (since it contains linear triples). The supposition that the quadruple is of the third kind leads immediately to a contradiction, for a quadruple with four linear triples (which are d-cyclic) and two points with a distance equal to d may be imbedded in a circle of metric diameter d, and hence is d-cyclic. Suppose, then, that the quadruple is of the second kind. Then the four points must form a convex tripod; *i. e.*, exactly three of the triples contained in the four points are linear, and one of the points lies between three pairs of points. But from the above two relations, p_2 lies between p_1 and p_3, while p_4 lies between p_1 and p_3. Hence, the four points do not form a convex tripod, and this case is also impossible. Hence, the lemma is proved.

LEMMA 2. *If a pseudo d-cyclic quintuple contains two diametral points, and a pseudo-linear quadruple which is pseudo d-cyclic, then the quintuple contains a convex tripod.*

Let p_1, p_2, p_3, p_4, p_5 be a quintuple satisfying the conditions of the lemma. We may choose the labeling so that p_2, p_3, p_4, p_5 is the pseudo-linear quadruple which is pseudo d-cyclic. Then the pair of diametral points that, by hypothesis, the quintuple contains, is not contained in this quadruple, since, by Lemma 1, a pseudo d-cyclic quadruple does not contain a pair of diametral points. We may letter the points so that p_1, p_5 are diametral. ,

Since the quadruple p_2, p_3, p_4, p_5 is pseudo-linear, we have $p_2p_3 = p_4p_5 = a$; $p_3p_4 = p_2p_5 = b$; $p_2p_4 = p_3p_5 = c$; and all four of the triples contained in the quadruple are linear. Then,

$$(a + b - c)(a - b + c)(-a + b + c) = 0.$$

Now, $p_1p_5 = d$, and hence we have $p_1p_2 + p_2p_5 = d$; $p_1p_4 + p_4p_5 = d$; $p_1p_3 + p_3p_5 = d$. Combining these relations with the ones above, we obtain $p_1p_2 = d - b$; $p_1p_3 = d - c$; $p_1p_4 = d - a$.

Consider, now the quadruple p_1, p_2, p_3, p_4. We have three cases to consider.

Case A. $a + b = c$. The point p_3 of the quadruple p_1, p_2, p_3, p_4 is seen to lie between the pairs p_1, p_2; p_1, p_4; p_2, p_4, while the triple p_1, p_2, p_4 is not linear. Thus, the quadruple forms a convex tripod.

Case B. $a + c = b$. The point p_2 lies between each of the three pairs of points p_1, p_3; p_1, p_4; p_3, p_4, and the quadruple p_1, p_2, p_3, p_4 is again seen to form a convex tripod.

Case C. $b + c = a$. Here, the point p_4 lies between three pairs of points and the quadruple is a convex tripod.

Thus, in any one of the three cases, the quintuple contains a convex tripod, and the lemma is proved.

These two lemmas enable us to prove the following theorem, of which much use is to be made:

THEOREM I. *If a pseudo d-cyclic quintuple contains a pair of diametral points, then the quintuple contains a convex tripod.*

Assume the labeling so that, as before, the points p_1, p_5 are diametral. We have, then, $p_1p_2 + p_2p_5 = d$; $p_1p_4 + p_4p_5 = d$; $p_1p_3 + p_3p_5 = d$. Now, since the circle has the congruence order four, at least one of the quadruples contained in the five points is pseudo d-cyclic. By Lemma 1, this quadruple does not contain the points p_1, p_5. We may choose the labeling so that the quadruple p_2, p_3, p_4, p_5 is pseudo d-cyclic. If this quadruple is pseudo-linear, the theorem is proved by an application of Lemma 2. Suppose, then, that this quadruple is not pseudo-linear. If it is a quadruple of the second kind, the theorem is proved. We suppose, finally, that the quadruple is of the first kind. Then we have the relations $p_2p_3 = p_4p_5 = a$; $p_3p_4 = p_2p_5 = b$; $p_2p_4 = p_3p_5 = c$; and $a + b + c = 2d$. Consider the quadruple p_1, p_2, p_3, p_4. It is seen at once that the point p_1 lies between each of the three pairs of points p_2, p_3; p_2, p_4; p_3, p_4, and the quadruple forms a convex tripod. Hence, the theorem is proved.

We give an example of a pseudo d-cyclic quintuple containing a pseudo-linear quadruple that is pseudo d-cyclic, a pair of diametral points, and a convex tripod. The ten distances determined by the five points are given by means of the table:

	p_1	p_2	p_3	p_4	p_5
p_1	0	$d/3$	$d/2$	$d/6$	d
p_2	$d/3$	0	$d/6$	$d/2$	$2d/3$
p_3	$d/2$	$d/6$	0	$d/3$	$d/2$
p_4	$d/6$	$d/2$	$d/3$	0	$5d/6$
p_5	d	$2d/3$	$d/2$	$5d/6$	0

This quintuple contains three d-cyclic quadruples. The pseudo-linear quadruple that is also pseudo d-cyclic is p_1, p_2, p_3, p_4, and the convex tripod is formed by the four points p_2, p_3, p_4, p_5.

3. We now establish certain lemmas concerning regular pseudo d-cyclic quintuples (that is, pseudo d-cyclic quintuples no four points of which form a convex tripod) that will enable us to prove the principal theorem of this paper characterizing regular pseudo d-cyclic sets containing more than four points.

LEMMA 1. *A regular pseudo d-cyclic quintuple does not contain exactly one d-cyclic quadruple.*

We suppose that the regular pseudo d-cyclic quintuple p_1, p_2, p_3, p_4, p_5 contains just a single d-cyclic quadruple. We may choose the labeling so that this quadruple is p_1, p_2, p_3, p_4. Then the remaining four quadruples are pseudo d-cyclic of the first or third kinds (since the quintuple is regular). In either case, if we let p_i, p_j, p_k, p_n represent any one of these quadruples, we have $p_ip_j = p_kp_n$; $p_ip_k = p_jp_n$; $p_jp_k = p_ip_n$. Writing these relations out for each of the four quadruples, it is seen that all of the ten distances determined by the five points are equal. Then, in particular, the quadruple p_1, p_2, p_3, p_4 is equilateral, and hence is not d-cyclic, contrary to the hypothesis. Thus, the lemma is proved.

LEMMA 2. *A regular pseudo d-cyclic quintuple does not contain exactly two d-cyclic quadruples.*

We suppose that the regular pseudo d-cyclic quintuple p_1, p_2, p_3, p_4, p_5 contains two (exactly) d-cyclic quadruples, and we select the labeling of the points so that these two d-cyclic quadruples are p_1, p_2, p_3, p_4 and p_2, p_3, p_4, p_5. Then the remaining three quadruples are pseudo d-cyclic of the first or third kinds, and, in either case, we have the following relations:

$$\text{(A)} \quad p_1p_2 = p_1p_3 = p_1p_4 = p_2p_5 = p_3p_5 = p_4p_5; \quad p_2p_3 = p_2p_4 = p_3p_4 = p_1p_5.$$

Since the quadruple p_1, p_2, p_3, p_4 is d-cyclic, it contains at least two linear triples. Now the triple p_2, p_3, p_4 is equilateral, and hence, it is not linear. Then two of the remaining three triples, p_1, p_2, p_3; p_1, p_2, p_4; p_1, p_3, p_4 are linear. We have, thus, three cases to consider. All three cases are treated similarly to the one discussed here.

Case A. The triples p_1, p_2, p_3; p_1, p_2, p_4 are linear. Then relations (A) yield that $p_2p_1p_3$ and $p_2p_1p_4$ exist. But then, we have $p_3p_1p_4$, and the four points form a convex tripod, and hence are not d-cyclic, contrary to the hypothesis.

Thus, the lemma is proved.

LEMMA 3. *A regular pseudo d-cyclic quintuple does not contain exactly three d-cyclic quadruples.*

We suppose the contrary, and select the labeling so that p_1, p_2, p_3, p_4; p_2, p_3, p_4, p_5 are the two pseudo d-cyclic quadruples. Then we have the relations $p_1p_2 = p_3p_4 = p_2p_5$; $p_1p_3 = p_2p_4 = p_3p_5$; $p_2p_3 = p_1p_4 = p_4p_5$. We distinguish two cases:

Case A. The pseudo d-cyclic quadruples p_1, p_2, p_3, p_4; p_2, p_3, p_4, p_5 are both of the first kind. The three quadruples p_1, p_2, p_3, p_5; p_1, p_2, p_4, p_5; p_1, p_3, p_4, p_5 are, by hypothesis, d-cyclic, and hence each contains at least two linear triples. Since the two quadruples p_1, p_2, p_3, p_4; p_2, p_3, p_4, p_5 are supposed pseudo d-cyclic of the first kind, none of the seven distinct triples that they contain are linear. Thus, of the ten triples contained in the five points, only three of them are linear. These triples are p_1, p_2, p_5; p_1, p_3, p_5; p_1, p_4, p_5. Each of the three d-cyclic quadruples must contain two of these triples. From the above relations, we obtain that $p_1p_2p_5$; $p_1p_3p_5$; $p_1p_4p_5$ exist; *i. e.*, $p_1p_5 = 2(p_1p_2) = 2(p_1p_3) = 2(p_1p_4)$. Then $p_1p_2 = p_2p_3 = p_1p_3$, and since the triple p_1, p_2, p_3 is d-cyclic, each of these distances equals $2d/3$. But then $p_1p_5 = 4d/3$, which is impossible, for since any triple containing p_1, p_5 is d-cyclic, the distance p_1p_5 cannot exceed d. Hence this case is not possible.

Case B. One of the quadruples p_1, p_2, p_3, p_4; p_2, p_3, p_4, p_5 is of the third kind. Then both of the quadruples are of the third kind; for, since the quintuple is regular, pseudo d-cyclic quadruples are either of the first or third kinds. The triple p_2, p_3, p_4 is common to both of the quadruples, and the supposition that one quadruple is of the third kind means that this triple is linear. Then the other quadruple contains a linear triple, and hence it is of the third kind. Then the seven distinct triples contained in these two quadruples are linear, and in addition, we have, as before, the relations $p_1p_2 = p_3p_4 = p_2p_5$; $p_1p_3 = p_2p_4 = p_3p_5$; $p_2p_3 = p_1p_4 = p_4p_5$. Now, we may label the three points common to the two quadruples so that $p_2p_3p_4$ exists. Applying the above relations, we have $p_1p_2p_3$; $p_3p_4p_1$; $p_4p_1p_2$; $p_3p_4p_5$; $p_4p_5p_2$; $p_3p_2p_5$ holding. Consider the d-cyclic quadruple p_1, p_2, p_3, p_5. Two of its triples are linear in the order $p_1p_2p_3$ and $p_3p_2p_5$. Three sub-cases present themselves.

Sub-case 1. The remaining two triples contained in p_1, p_2, p_3, p_5 are not linear. Since these two triples are d-cyclic we have $p_1p_3 + p_3p_5 + p_5p_1 = 2d$, $p_1p_2 + p_2p_5 + p_5p_1 = 2d$. Subtracting the second relation from the first,

and making use of the fact that $p_3p_2p_5$ exists, we obtain $p_1p_3 = p_1p_2 - p_2p_3$, which is impossible, since $p_1p_2p_3$ exists. Hence, this sub-case cannot occur.

Sub-case 2. Only one of the two triples p_1, p_3, p_5; p_1, p_2, p_5 is linear. Suppose that p_1, p_3, p_5 is linear. Then $p_1p_3p_5$ exists, and since p_1, p_2, p_5 is not linear, we have $p_1p_2 + p_2p_5 + p_5p_1 = 2d$. From these relations, it is easily seen that $p_1p_5 = d + p_2p_3$, which is impossible. Hence this cannot occur.

Suppose that p_1, p_2, p_5 is selected as the only one of the two triples to be linear. Then we obtain $p_1p_2p_5$, and the point p_2 is seen to lie between p_1p_5; p_1p_3; p_3p_5; *i. e.*, the points p_1, p_2, p_3, p_5 form a convex tripod and hence are not d-cyclic, as supposed.

Sub-case 3. Both of the triples p_1, p_2, p_5; p_1, p_3, p_5 are linear. Then $p_1p_2p_3$; $p_3p_2p_5$; $p_1p_3p_5$; $p_1p_2p_5$ exist. From these relations, it is immediate that $p_1p_2 = p_1p_3$, which is impossible, for $p_1p_2p_3$ exists. A contradiction in the form $p_1p_2 = p_1p_3 + p_3p_2$ may also be found.

Hence, none of the three sub-cases under Case B can occur, and the theorem is proved.

LEMMA 4. *A regular pseudo d-cyclic quintuple does not contain exactly one pseudo d-cyclic quadruple.*

The proof of this final lemma is divided into two parts.

Part 1. We suppose that the regular pseudo d-cyclic quintuple, p_1, p_2, p_3, p_4, p_5 contains exactly one pseudo d-cyclic quadruple, and we assume that this quadruple is of the first kind. We select the labeling so that this quadruple is p_1, p_2, p_3, p_4. Then none of the four triples contained in this quadruple is linear. The quadruples p_1, p_2, p_3, p_5; p_1, p_2, p_4, p_5; p_1, p_3, p_4, p_5; p_2, p_3, p_4, p_5 are each d-cyclic and hence each contains at least two linear triples. Since each of these quadruples contains a non-linear triple from the quadruple p_1, p_2, p_3, p_4, they can contain at most three linear triples. But if a d-cyclic quadruple contains three linear triples, while the fourth triple is not linear, the quadruple must contain two diametral points.* But then, according to the theorem of section 2, the quintuple would contain a convex tripod, which is impossible, since the quintuple is, by hypothesis, regular. Hence, each of the four d-cyclic quadruples contains exactly two linear triples. We may select the triples p_1, p_3, p_5; p_2, p_3, p_5 in the quadruple p_1, p_2, p_3, p_5 to be linear. Then, in order that one of the remaining three quadruples may not

* L. M. Blumenthal, "A Complete Characterization of Proper Pseudo D-Cyclic Sets of Points," *American Journal of Mathematics*, Vol. 54 (1932), p. 393.

have three linear triples, we must select p_1, p_4, p_5; p_2, p_4, p_5 as linear. Then each of the four d-cyclic quadruples contains exactly two linear triples. The triples p_1, p_2, p_5; p_3, p_4, p_5 are not linear, but since they are d-cyclic, we have the relations $p_1p_2 + p_2p_5 + p_5p_1 = 2d$; $p_3p_4 + p_4p_5 + p_5p_3 = 2d$. Also, from the fact that the quadruple p_1, p_2, p_3, p_4 is pseudo d-cyclic, we have $p_1p_2 = p_3p_4$; $p_1p_4 = p_2p_3$; $p_1p_3 = p_2p_4$. From the linearity of the four triples, we have the relations:

$$
\text{(A)}\quad
\begin{aligned}
&(p_1p_3 + p_3p_5 - p_5p_1)(p_1p_3 - p_3p_5 + p_5p_1)(-p_1p_3 + p_3p_5 + p_5p_1) = 0\\
&(p_2p_3 + p_3p_5 - p_5p_2)(p_2p_3 - p_3p_5 + p_5p_2)(-p_2p_3 + p_3p_5 + p_5p_2) = 0\\
&(p_1p_4 + p_4p_5 - p_5p_1)(p_1p_4 - p_4p_5 + p_5p_1)(-p_1p_4 + p_4p_5 + p_5p_1) = 0\\
&(p_2p_4 + p_4p_5 - p_5p_2)(p_2p_4 - p_4p_5 + p_5p_2)(-p_2p_4 + p_4p_5 + p_5p_2) = 0
\end{aligned}
$$

An examination of the first two of the relations (A) together with the relations that immediately precede them, leads to the result that either $p_1p_3p_5$ and $p_3p_5p_2$ or $p_1p_5p_3$ and $p_2p_3p_5$ exist. A similar examination of the last two of the relations (A) yields the fact that either $p_1p_4p_5$ and $p_4p_5p_2$ or $p_1p_5p_4$ and $p_2p_4p_5$ exist. But it is easily shown that no one of the four cases obtained by grouping these two pairs of relations is possible. Hence the case considered by Part 1 of the theorem cannot occur.*

Part 2. In this part we assume that the regular pseudo d-cyclic quintuple p_1, p_2, p_3, p_4, p_5 contains exactly one pseudo d-cyclic quadruple, and this quadruple is pseudo-linear. We may label the points so that the quadruple p_1, p_2, p_3, p_4 is pseudo-linear, and so that $p_1p_2p_3$, $p_2p_3p_4$, $p_3p_4p_1$, $p_4p_1p_2$ exist.

Consider, now, any one of the four d-cyclic quadruples, say p_1, p_2, p_4, p_5

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* To see, for example, that of the nine combinations that may be obtained from the first two of the relations (A) only the two combinations given above are consistent with the relations preceding (A), it is sufficient to examine each of these nine combinations. To indicate the method employed to reject combinations, we consider the case of $p_1p_3p_5$, $p_2p_3p_5$; i. e.,

(1) $$p_1p_3 + p_3p_5 = p_1p_5,$$

(2) $$p_2p_3 + p_3p_5 = p_2p_5.$$

From these two relations we obtain

(3) $$p_1p_5 - p_2p_5 = p_1p_3 - p_2p_3.$$

Now the triples p_1, p_2, p_3; p_1, p_2, p_5 are d-cyclic, *not* linear. Whence, we have

(4) $$p_1p_2 + p_2p_3 + p_3p_1 = 2d,$$

(5) $$p_1p_2 + p_2p_5 + p_5p_1 = 2d.$$

From (3) and (4) we obtain, $2(p_1p_3) = 2d - p_1p_2 - p_2p_5 + p_1p_5$, and using (5), we get $2(p_1p_3) = 2(p_1p_5)$. But this, together with (1) implies that $p_3p_5 = 0$, which is impossible, since the points p_3 and p_5 are distinct.

contained in the quintuple. This quadruple being d-cyclic, and not containing two diametral points (by reason of Lemma 2, section 2), must contain either exactly two or exactly four linear triples. Two cases present themselves.

Case A. The quadruple p_1, p_2, p_4, p_5 contains four linear triples. Consider the d-cyclic quadruple p_2, p_3, p_4, p_5. This quadruple contains the two linear triples p_2, p_3, p_4 and p_2, p_4, p_5. But, since $p_2p_3p_4$ exists, the linearity of p_2, p_4, p_5 in any order is possible only if all four of the triples contained in this quadruple are linear. Then evidently all ten of the triples contained in the quintuple are linear. In a similar manner, this result is seen to follow if any of the four d-cyclic quadruples is assumed to have four linear triples.

Case B. Each of the four d-cyclic quadruples has exactly two linear triples. The quadruples p_1, p_2, p_4, p_5; p_1, p_3, p_4, p_5; p_1, p_2, p_3, p_5; p_2, p_3, p_4, p_5 contain the linear triples p_1, p_2, p_4; p_1, p_3, p_4; p_1, p_2, p_3; p_2, p_3, p_4, respectively, and each contains exactly one other linear triple. It is found that this is possible only if one of the three following combinations are selected as linear:

(1) p_1, p_2, p_5; p_3, p_4, p_5 (2) p_1, p_3, p_5; p_2, p_4, p_5 (3) p_2, p_3, p_5; p_1, p_4, p_5.

We examine the first combination. The quadruple p_1, p_2, p_3, p_5 contains only the two linear triples p_1, p_2, p_3 and p_1, p_2, p_5. Since $p_1p_2p_3$ exists, the second triple is linear in the order $p_2p_1p_5$. Consider, now, the quadruple p_1, p_2, p_4, p_5. It contains the two linear triples p_1, p_2, p_4; p_1, p_2, p_5, and since $p_2p_1p_4$ and $p_2p_1p_5$ exist, all of the triples contained in this quadruple are linear, contrary to the hypothesis for Case B. Hence combination (1) cannot occur.

To show the impossibility of combination (2), consider the quadruple p_1, p_2, p_3, p_5. Since $p_1p_2p_3$ exists, the linearity of p_1, p_3, p_5 in any order demands the linearity of all four of its triples.

Finally, we consider the combination (3). From the quadruple p_1, p_3, p_4, p_5 we obtain $p_3p_4p_1$ and $p_4p_1p_5$. Then the quadruple p_1, p_2, p_4, p_5 is seen to have all of its triples linear, since $p_2p_1p_4$ and $p_4p_1p_5$ exist. Thus, the assumption that each of the quadruples contained in the quintuple has exactly two linear triples leads to a contradiction. Hence, Case A alone is possible, and all ten triples contained in the five points are linear.

But the straight line is known to have the quasi-congruence order 3 *; *i. e.*, any set of points containing more than four points such that each triple is congruent to three points of a line, is congruent to a sub-set of the line.

* Karl Menger, "New Foundation of Euclidean Geometry," *American Journal of Mathematics*, Vol. 53 (1931), p. 727.

The five points p_1, p_2, p_3, p_4, p_5 have been shown to have all of their triples linear, and hence they are congruent with five points of a line. This, however, is impossible, for the five points contain the quadruple p_1, p_2, p_3, p_4, which, by hypothesis is pseudo-linear. Hence the five points cannot be congruent with five points of a line.

Thus, Case A having been shown to be impossible, the theorem is proved.

Since the circle has the congruence order 4, all of the quadruples contained in a pseudo d-cyclic quintuple cannot be d-cyclic. This fact, together with the four lemmas proved above, establishes the following theorem:

THEOREM II. *A regular pseudo d-cyclic quintuple does not contain any d-cyclic quadruples.*

Hence, if a set of five points is such that (1) all ten of its triples are d-cyclic, (2) the set does not contain a convex tripod, (3) one of its quadruples is d-cyclic, then the set is congruent with five points of a circle of metric diameter d.

We have immediately the theorem characterizing regular pseudo d-cyclic quintuples.

THEOREM III. *A regular pseudo d-cyclic quintuple is equilateral.*

Since the quintuple does not contain any d-cyclic quadruples, all of its quadruples are pseudo d-cyclic of either the first or third kinds. In either case, the quadruple has its "opposite" distances equal. Writing these relations for each of the five quadruples, we have immediately that all ten of the distances determined by the five points are equal. Hence the quintuple is equilateral. Since each triple is d-cyclic, we note that each of the ten distances equals $2d/3$.

COROLLARY 1. *All quadruples of a regular pseudo d-cyclic quintuple are pseudo d-cyclic of the first kind.*

COROLLARY 2. *A regular pseudo d-cyclic quintuple does not contain any linear triples.*

4. We consider now regular pseudo d-cyclic sets containing more than four points. For these sets we prove the following theorem.

THEOREM IV. *A regular pseudo d-cyclic set containing more than four points is equilateral.*

We have seen that this theorem is true if the set consists of exactly five points. To prove the theorem, we shall assume it true for a set consisting of

k points ($k > 4$), and show that it follows that it is true for sets containing $k+1$ points.

Let $p_1, p_2, \cdots, p_k, p_{k+1}$ be a regular pseudo d-cyclic set containing exactly $k+1$ points. This set contains at least one regular pseudo d-cyclic set of k points; for, otherwise, every set of k points contained in the $k+1$ points would be d-cyclic, and since $k > 4$, every quadruple contained in the $k+1$ points would be d-cyclic, *a fortiori*. But, since the circle has the congruence order 4, the $k+1$ points would be d-cyclic, instead of pseudo d-cyclic, as supposed.

We may assume the labeling so that $p_1, p_2, \cdots, p_k$ is pseudo d-cyclic. We now show that at least one other set of k points contained in the $k+1$ points is pseudo d-cyclic. Suppose that each of the other k sets of k points contained in the $k+1$ points is d-cyclic. Then all of the quadruples contained in these k sets are d-cyclic. But all of the quadruples to be found in the pseudo d-cyclic set of k points, $p_1, p_2, \cdots, p_k$, are contained in the remaining k sets, and since these quadruples are all d-cyclic, the k points $p_1, p_2, \cdots, p_k$ have all their quadruples d-cyclic and hence are themselves d-cyclic, which contradicts the previous assumption. Hence the $k+1$ points contain another pseudo d-cyclic set of k points. We may select the labeling so that this set is $p_2, p_3, \cdots, p_k, p_{k+1}$. The two pseudo d-cyclic sets of k points shown to be contained in the $k+1$ points are regular, and since the theorem is assumed true for regular pseudo d-cyclic sets of k points, we have that each of these two sets is equilateral, with all of their distances equal to $2d/3$. Thus, of the $(1/2)k(k+1)$ distances determined by the $k+1$ points, all are seen to be equal except the distance p_1p_{k+1}, which does not enter into the above two sets. To determine this distance, consider any triple, say p_1, p_2, p_{k+1}, containing this pair of points. This triple is not linear, for if it were, then

$$(p_1p_2 + p_2p_{k+1} - p_1p_{k+1})(p_1p_2 - p_2p_{k+1} + p_1p_{k+1})(-p_1p_2 + p_2p_{k+1} + p_1p_{k+1}) = 0,$$

and since $p_1p_2 = p_2p_{k+1} = 2d/3$, we have $p_1p_{k+1} = 4d/3$, which is impossible. Hence the triple p_1, p_2, p_{k+1} is not linear, and since it is d-cyclic, we have $p_1p_2 + p_2p_{k+1} + p_1p_{k+1} = 2d$, from which $p_1p_{k+1} = 2d/3$. Thus, the set of $k+1$ points is equilateral, and the theorem is proved.

Two corollaries similar to those following the theorem characterizing regular pseudo d-cyclic quintuples may be stated.

THE RICE INSTITUTE.

ON ARRAYS OF NUMBERS.*

By LEONARD CARLITZ.†

1. *Introduction.* This paper is concerned with the investigation of arrays of numbers, $a_{n,s}$, defined by a relation of the type

$$a_{n+1,s} = \sum_{i=0}^{N} \beta_i(s, n) a_{n,s-N+i}, \tag{1}$$

—where N is independent of both s and n,—plus a set of "initial" conditions such as

$$a_{1,1} = 1; \qquad a_{1,s} = 0 \quad \text{for} \quad s \neq 1.‡$$

Thus, for example, for $N = 1$, $\beta_0 \equiv \beta_1 \equiv 1$, we have the Pascal Triangle—that is, the array of combinatorial coefficients; for $N = 1$, $\beta_0 \equiv 1$, $\beta_1 = s$, the array of Stirling numbers.

The numbers $a_{n,s}$ are best studied by means of the operator of order N,

$$\beta_0 E^N + \beta_1 E^{N-1} + \cdots + \beta_N, \tag{2}$$

where E is the well known symbol defined by

$$Ef(s) = f(s-1);$$

we suppose in the following that E operates only on s. We shall be interested in determining "explicit" expressions for the elements of the array (1). As will appear in § 6 fairly simple expressions may be obtained when the operator (2) satisfies certain conditions. Before deriving actual expressions for the constituents of an arbitrary array, we first set up certain arrays that seem to possess some interest in themselves (§§ 2, 3, 4).§

In § 7 some of the results deduced in the earlier parts of the paper are applied to the evaluation of certain finite sums. The consideration of one of these sums indicates a connection between particular arrays and Bernoulli polynomials in several variables.

2. *Expansion of* $(x^{\lambda+\mu}D^{\mu})^n$. It is evident that we may put (μ integer $\geqq 0$)

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‡ Precisely these conditions hold in all cases studied in the present paper.

§ A number of special cases are treated by the writer in the *American Mathematical Monthly*, Vol. 37 (1930), pp. 472-479.

$$(3) \qquad (x^{\lambda+\mu}D^{\mu})^n = \sum_{s=1}^{\mu(n-1)+1} a_{n,s}(x^{s+\mu-1+n\lambda}D^{s+\mu-1}),$$

where of course $D \equiv d/dx$. In order to determine the $a_{n,s}$ we apply $(x^{\lambda+\mu}D^{\mu})$ to both sides of (3). Then

$$(x^{\lambda+\mu}D^{\mu})^{n+1} = \sum a_{n,s}(x^{\lambda+\mu}D^{\mu})(x^{s+\mu-1+n\lambda}D^{s+\mu-1})$$

$$= \sum_s a_{n,s} \sum_{j=0}^{\mu} \binom{\mu}{j}(s+\mu-1+n\lambda)!^j x^{s+2\mu-j-1+(n+1)\lambda}D^{s+2\mu-j-1},$$

where

$$\binom{\mu}{j} = \frac{\mu!}{j!(\mu-j)!}; \qquad \mu!^j = \mu(\mu-1)\cdots(\mu-j+1), \quad \mu!^0 = 1.$$

Writing t for $s+\mu-j$,

$$(x^{\lambda+\mu}D^{\mu})^{n+1} = \sum_{t=1}^{\mu n+1} \sum_{j=0}^{\mu} a_{n,t-\mu+j}\binom{\mu}{j}(t+j-1+n\lambda)!^j(x^{t+\mu-1+(n+1)\lambda}D^{t+\mu-1}).$$

Comparison with (3) shows that

$$(4) \qquad a_{n+1,s} = \sum_{j=0}^{\mu}\binom{\mu}{j}(s+j-1+n\lambda)!^j a_{n,s-\mu+j}$$
$$= \sum \binom{\mu}{j}(s+j-1+n\lambda)!^j E^{\mu-j}\cdot a_{n,s}.$$

Evidently the $a_{n,s}$ satisfy the initial conditions of § 1 and hence define an array of numbers of the type to be considered. The associated operator is

$$\sum_j \binom{\mu}{j}(s+j-1+n\lambda)!^j E^{\mu-j};$$

it is easily verified that this can be written

$$(5) \qquad = (E+s+n\lambda)(E+s+1+n\lambda)\cdots(E+s+\mu-1+n\lambda),$$

and that the operators in (5) are all commutative.

3. *Expansion of* $(x^{\mu}D^{\lambda+\mu})^n$. We suppose now that λ and μ are both non-negative integers. By the results of the preceding section the expansion of the operator $(x^{\mu}D^{\lambda+\mu})^n$ would appear to require an array of *order* (i. e. the order of the associated operator) $\lambda+\mu$; however it is not difficult to show that an array of order μ will suffice. Accordingly we put

$$(6) \qquad (x^{\mu}D^{\lambda+\mu})^n = \sum_{s=1}^{\mu(n-1)+1} b_{n,s}(x^{s+\mu-1}D^{s+\mu-1+n\lambda}).$$

As above apply $x^{\mu}D^{\lambda+\mu}$ to both sides of (6):

$$(x^{\mu}D^{\lambda+\mu})^{n+1} = \sum^{s} b_{n,s}\sum_{j=0}^{\lambda+\mu}\binom{\lambda+\mu}{j}(s+\mu-1)!^j(x^{s+2\mu-j-1}D^{s+2\mu-j-1+(n+1)\lambda})$$

$$= \sum_t\sum_j \binom{\lambda+\mu}{j}(t+j-1)!^j b_{n,t-\mu+j}(x^{t+\mu-1}D^{t+\mu-1+(n+1)\lambda}).$$

Comparing with (6),

$$b_{n+1,s} = \sum_{j=0}^{\lambda+\mu} \binom{\lambda+\mu}{j} (s+j-1)\,!^{\,j}\, b_{n,s-\mu+j} \tag{7}$$

$$= \sum \binom{\lambda+\mu}{j} (s+j-1)\,!^{\,j}\, E^{\mu-j}\cdot b_{n,s}.$$

However this is not of the form (1); and indeed it involves an operator of order $\lambda+\mu$. In order to attain the desired result we notice first that the operator in (7)

$$= (E+s)(E+s+1)\cdots(E+s+\lambda+\mu-1)E^{-\lambda},$$

where as in (5) the factors are commutative. Then

$$b_{n+1,s} = (E+s)\cdots(E+s+\lambda+\mu-1)E^{-\lambda}b_{n,s}$$

$$= \prod_{h=0}^{\lambda+\mu-1}(E+s+h)\cdot E^{-\lambda}\prod_{k=0}^{\lambda+\mu-1}(E+s+k)\cdot E^{-\lambda}b_{n-1,s}$$

$$= \prod(E+s+h)\cdot\prod(E+s+\lambda+k)\cdot E^{-2\lambda}b_{n-1,s}$$

$$= \prod_{h=0}^{\lambda+\mu-1}\prod_{k=0}^{n-1}(E+s+h+k\lambda)\cdot E^{-n\lambda}b_{1,s}. \tag{8}$$

Now

$$\prod_{j=1}^{n\lambda}(E+s+j-1)\cdot E^{-n\lambda} = \sum_{j=0}^{n\lambda}\binom{n\lambda}{j}(s+j-1)\,!^{\,j}\,E^{-j};$$

and putting

$$\prod_{h=0}^{\mu-1}\prod_{k=1}^{n}(E+s+h+k\lambda) = \sum_{h=0}^{n\mu}A_hE^h$$

(8) becomes

$$\sum_{h=0}^{n\mu}\sum_{j=0}^{n\lambda}\binom{n\lambda}{j}A_h(s+j-h-1)\,!^{\,j}\,b_{1,s+j-h}.$$

But $b_{1,s+j-h}$ vanishes except when $s+j-h=1$, and then $(s+j-h-1)\,!^{\,j}$ vanishes unless $j=0$. Therefore

$$b_{n+1,s} = \prod_{h=0}^{\mu-1}\prod_{k=1}^{n}(E+s+h+k\lambda)\cdot b_{1,s},$$

and finally

$$b_{n+1,s} = \prod_{h=1}^{\mu}(E+s+h-1+n\lambda)\cdot b_{n,s}, \tag{9}$$

which defines an array of order μ; indeed it is identical with that defined by (4) of § 2.

4. *Arrays associated with*

$$m\,!^{\,\lambda+\mu}(m-\lambda)\,!^{\,\lambda+\mu}\cdots(m-[n-1]\,\lambda)\,!^{\,\lambda+\mu}.$$

We begin with the proof of the identity

$$(10)\quad m\,!^{\lambda+\mu}(m-n\lambda)\,!^{\lambda+\mu} = \sum_{j=0}^{\lambda+\mu} \Lambda_j(k,n)(m+k+j-1)\,!^{j}(m-[n+1]\lambda-\mu-k+1)\,!^{j}$$

where

$$\Lambda_j(k)=\Lambda_j(k,n) = \binom{\lambda+\mu}{j}(k+\lambda+\mu-1)\,!^{\lambda+\mu-j}(k+[n+1]\lambda+\mu-1)\,!^{\lambda+\mu-j}$$

for all integral $k \geqq 1-\lambda-\mu$.

Evidently (10) holds for $k=1-\lambda-\mu$; assuming then that it holds for all larger values up to and including k, we show without difficulty that it holds for $k+1$. Since

$$\begin{aligned}(m+k)(m-[n+1]\lambda-\mu-k+1) &= (m+k+j)(m-[n+1]\lambda-\mu-k-j-1)\\ &\quad + j([n+1]\lambda+\mu+2k+j-1),\end{aligned}$$

the right side of (10)

$$\begin{aligned}&= \sum_{j=0}^{\lambda+\mu}\Lambda_j(k)\Big[(m+k+j)\,!^{j}(m-[n+1]\lambda-\mu-k)\,!^{j}\\ &+ j([n+1]\lambda+\mu+2k+j-1)(m+k+j-1)\,!^{j-1}(m-[n+1]\lambda-\mu-k)\,!^{j-1}\\ &= \sum_{j=0}^{\lambda+\mu}\{\Lambda_j(k)+(j+1)([n+1]\lambda+\mu+2k+j)\Lambda_{j+1}(k)\}\\ &\qquad \times (m+k+j)\,!^{j}(m-[n+1]\lambda-\mu-k)\,!^{j}.\end{aligned}$$

But the quantity within the { }

$$\begin{aligned}&= \binom{\lambda+\mu}{j}(k+\lambda+\mu-1)\,!^{\lambda+\mu-j}(k+[n+1]\lambda+\mu-1)\,!^{\lambda+\mu-j}\\ &\quad +(j+1)([n+1]\lambda+\mu+2k+j)\binom{\lambda+\mu}{j+1}(k+\lambda+\mu-1)\,!^{\lambda+\mu-j-1}\\ &\qquad \times (k+[n+1]\lambda+\mu-1)\,!^{\lambda+\mu-j-1}\\ &= \binom{\lambda+\mu}{j}(k+\lambda+\mu-1)\,!^{\lambda+\mu-j-1}(k+[n+1]\lambda+\mu-1)\,!^{\lambda+\mu-j-1}\\ &\qquad \cdot\big[(k+j)(k+j+n\lambda)+(\lambda+\mu-j)([n+1]\lambda+\mu+2k+j)\big]\\ &= \binom{\lambda+\mu}{j}(k+\lambda+\mu)\,!^{\lambda+\mu-j}(k+[n+1]\lambda+\mu)\,!^{\lambda+\mu-j}\end{aligned}$$

completing the induction.

4.1. We now consider the expansion

$$(11)\ldots\quad m\,!^{\lambda+\mu}(m-\lambda)\,!^{\lambda+\mu}\cdots(m-2(n-1)\lambda)\,!^{\lambda+\mu} = \sum_{s=1}^{\mu n+1} c_{n,s}(m+s-1)\,!^{(2n-1)\lambda+2s+\mu-2};$$

that such an expansion exists may be shown by repeated application of the obvious identity

$$m(m-h)=(m+k)(m-h-k)+(h+k)k.$$

To determine the $c_{n,s}$, put $m-\lambda$ in place of m in (11); and then multiply the left side of (11) by

$$m\,!^{\lambda+\mu}(m-2n\lambda)\,!^{\lambda+\mu} \tag{12}$$

and the right side by the expression obtained by applying (10) to (12):

$$m\,!^{\lambda+\mu}\cdots(m-2n\lambda)\,!^{\lambda+\mu}=\sum_{s=1}^{\mu(n-1)+1} c_{n,s}(m+s-\lambda-1)\,!^{(2n-1)\lambda+2s+\mu-2}$$
$$\cdot\sum_{j=0}^{\lambda+\mu}\Lambda_j(s-\lambda)\cdot(m+s-\lambda+j-1)\,!^{j}(m-2n\lambda-\mu-s+1)\,!^{j}$$
$$=\sum_s\sum_j\Lambda_j(s-\lambda,2n)c_{n,s}(m+s-\lambda+j-1)\,!^{(2n-1)\lambda+2s+2j+\mu-2}$$
$$=\sum_{t=1}^{\mu n+1}\sum_{j=0}^{\lambda+\mu}\Lambda_j(t-j,2n)c_{n,t-j+\lambda}(m+t-1)\,!^{(2n+1)\lambda+2t+\mu-2}.$$

Comparison with (11) shows that

$$c_{n+1,s}=\sum\Lambda_j(s-j,2n)c_{n,s-j+\lambda} \tag{13}$$
$$=\sum_{j=0}^{\lambda+\mu}\Lambda_j(s-j,2n)E^{j-\lambda}\cdot c_{n,s}.$$

We now proceed exactly as at the corresponding point in § 3; we may prove by an easy induction

$$\sum_{j=0}^{\lambda+\mu}\Lambda_j(s-j,2n)E^j=\prod_{j=1}^{\lambda+\mu}\Big[E+(s+j-1)(s+[2n+1]\lambda+\mu-j)\Big],$$

the factors on the right being permutable.* (13) then becomes

$$c_{n+1,s}=\prod_{j=1}^{\lambda+\mu}\Big[E+(s+j-1)(s+[2n+1]\lambda+\mu-j)\Big]E^{-\lambda}\cdot c_{n,s}$$
$$=\prod_{j=1}^{\lambda+\mu}\Big[E+(s+j-1)(s+[2n+1]\lambda+\mu-j)\Big]$$
$$\cdot\prod_{i=1}^{\lambda+\mu}[E+(s+\lambda+i-1)(s+2n\lambda+\mu-i)]\cdot E^{-2\lambda}\cdot c_{n-1,s}$$

* It will be noticed that in the following factorization of the same operator the factors are *not* permutable:

$$(E+s(s+2n\lambda))\,(E+(s+1)(s+2n\lambda+1))$$
$$\cdots(E+(s+\lambda+\mu-1)(s+[2n+1]\lambda+\mu-1)).$$

$$(14)\quad = \prod_{j=1}^{\lambda+\mu} \prod_{h=0}^{n-1} \Big[E + (s + h\lambda + j - 1)(s + [2n - h + 1]\lambda + \mu - j) \Big] \cdot E^{-n\lambda} \cdot c_{1,s}.$$

But

$$\prod_{j=1}^{n\mu} \Big[E + (s + j - 1)(s + [2n + 1]\lambda + \mu - j) \Big] \cdot E^{-n\lambda}$$

$$= \sum_{j=0}^{n\mu} \binom{n\lambda}{j} (s + j - 1)\,!^{j} (s + [n + 1]\lambda + \mu + j - 1)\,!^{j} E^{-j},$$

and, if we put

$$\prod_{j=1}^{\mu} \prod_{h=1}^{n} \Big[E + (s + h\lambda + j - 1)(s + [2n - h + 1]\lambda + \mu - j) \Big] = \sum_{h=0}^{\mu n} A_h E^h,$$

(14) becomes

$$\sum_{h,j} A_h \binom{n\lambda}{j} (s - h + j - 1)\,!^{j} (s + [n + 1]\lambda + \mu + j - h - 1)\,!^{j} E^{h-j} \cdot c_{1,s},$$

so that all terms vanish except those for which $s - h + j = 1$; and the presence of $(s - h + j - 1)\,!^{j}$ necessitates $j = 0$. Finally, therefore, (14) may be rewritten as

$$(15)\quad c_{n+1,s} = \prod_{j=1}^{\mu} \prod_{h=1}^{n} \Big[E + (s + h + j - 1)(s + [2n - h + 1]\lambda + \mu - j) \Big] \cdot c_{1,s};$$

while this does not reduce to (1), yet as will appear from the results of the next section, from this form an explicit expression for the $c_{n,s}$ is easily obtained (see (36)).

4.2. If we assume μ to be even, it is possible to replace (11) by an expression holding when the number of factorials in the left member of (11) is either odd or even. This expansion is (writing 2ν for μ)

$$(16)\quad m\,!^{\lambda+2\nu} \cdots (m - [n - 1]\lambda)\,!^{\lambda+2\nu} = \sum_{s=1}^{\nu(n-1)+1} d_{n,s}(m + s - 1)\,!^{2(s+\nu-1)+n\lambda}.$$

For n odd this reduces to (11); assume then $n = 2p$. We proceed exactly as above. In (16) put $m - \lambda$ in place of m, multiply the two sides of the resulting equality by the corresponding members of (applying (10))

$$m\,!^{\lambda+2\nu}(m - [2p + 1]\lambda)\,!^{\lambda+2\nu}$$

$$= \sum_{j=0}^{\lambda+2\nu} \Lambda_j(k, 2p + 1)(m + k + j - 1)\,!^{j}(m - [2p + 2]\lambda - 2\nu - k + 1)\,!^{j}.$$

Then without difficulty it is seen that

$$d_{2p+2,s} = \sum_{j=0}^{\lambda+2\nu} \binom{\lambda+2\nu}{j} (s+j-1)\,!^{\,j}(s+j-1+[2p+1]\lambda)\,!^{\,j}\, d_{2p,s-2\nu+j}$$

$$= \prod_{j=1}^{\lambda+2\nu} \Big[E + (s+j-1)(s+[2p+2]\lambda+2\nu-j) \Big] \cdot E^{-\lambda} \cdot d_{2p,s}$$

$$= \prod_{j=1}^{\lambda+2\nu} \Big[E + (s+j-1)(s+[2p+2]\lambda+2\nu-j) \Big]$$

$$\times \Big[E + (s+\lambda+j-1)(s+[2p+1]\lambda+2\nu-j) \Big] \cdot E^{-2\lambda} d_{2p-2,s}$$

$$(17) \qquad = \prod_{j=1}^{\lambda+2\nu} \prod_{h=0}^{p-1} \Big[E + (s+h\lambda+j-1) \times (s+[2p-h+2]\lambda+2\nu-j) \Big] \cdot E^{-p\lambda}\, d_{2,s}.$$

As for $d_{2,s}$, starting with the identity like (10),

$$(m-\lambda)\,!^{\,2\nu} = \sum_{i=0}^{\nu} \binom{\nu}{i} (\lambda+2\nu)\,!^{\,\nu-i}(\lambda+\nu)\,!^{\,\nu-i}(m+i)\,!^{\,i}(m-2\lambda-2\nu)\,!^{\,i},$$

we see that

$$m\,!^{\,\lambda+2\nu}(m-\lambda)\,!^{\,\lambda+2\nu} = \sum_{i=0}^{\nu} \binom{\nu}{i} (\lambda+2\nu)\,!^{\,\nu-i}(\lambda+\nu)\,!^{\,\nu-i}(m+i)\,!^{\,2i+2\lambda+2\nu},$$

whence

$$d_{2,j+1} = \binom{\nu}{j} (\lambda+2\nu)\,!^{\,\nu-j}(\lambda+\nu)\,!^{\,\nu-j};$$

or what amounts to the same thing,

$$d_{2,s} = \prod_{j=1}^{\nu} [E + (s+\lambda+j-1)(s+\lambda+2\nu-j)]\, d_{1,s}.$$

Substituting this into (17),

$$d_{2p+2,s} = \prod_{j=1}^{\lambda+2\nu} \prod_{h=0}^{p-1} \Big[E + (s+h\lambda+j-1)(s+[2p-h+2]\lambda+2\nu-j) \Big]$$

$$(18) \cdot \prod_{i=1}^{\nu} \Big[E + (s+[p+1]\lambda+i-1)(s+[p+1]\lambda+2\nu-i) \Big] \cdot E^{-p\lambda} d_{1,s}.$$

Noting that the brackets in the right of (18) are permutable, we may evidently apply the method of reduction already used several times. In this way we get

$$(19) \quad d_{2p,s} = \prod_{j=1}^{2\nu} \prod_{h=1}^{p-1} \Big[E + (s+h\lambda+j-1)(s+[2p-h]\lambda+2\nu-j) \Big]$$

$$\cdot \prod_{j=1}^{\nu} \Big[E + (s+p\lambda+j-1)(s+p\lambda+2\nu-j) \Big] \cdot d_{1,s}.$$

Now split the double product into two parts,

$$\prod_{j=1}^{2\nu}\prod_{h=1}^{p-1}=\prod_{j-1}^{\nu}\prod_{h}\cdot\prod_{j=\nu+1}^{2\nu}\prod_{h}=\Pi_1\cdot\Pi_2,$$

say; in the double product Π_2 replace j by $2\nu-j+1$, h by $2p-h$. Then

$$\Pi_1\Pi_2=\prod_{j=1}^{\nu}\prod_{\substack{h=1\\h\neq p}}^{2p-1}\Big[E+(s+h\lambda+j-1)(s+[2p-h]\lambda+2\nu-j)\Big],$$

and finally (19) becomes

$$(20)\quad d_{2p,s}=\prod_{j=1}^{\nu}\prod_{h=1}^{2p-1}\Big[E+(s+h\lambda+j-1)(s+[2p-h]\lambda+2\nu-j)\Big]\cdot d_{1,s}.$$

If n in (16) $=2p-1$, comparison with (11) shows that $d_{2p-1,s}=c_{p,s}$, which by (15)

$$=\prod_{j=1}^{2\nu}\prod_{h=1}^{p-1}\Big[E+(s+h\lambda+j-1)(s+[2p-h-1]\lambda+2\nu-j)\Big]\cdot c_{1,s},$$

and this, exactly as (19) was transformed into (20), becomes

$$(21)\quad d_{2p-1,s}=\prod_{j=1}^{\nu}\prod_{h=1}^{2p-2}\Big[E+(s+h\lambda+j-1)(s+[2p-h-1]+2\nu-j)\Big]\cdot d_{1,s}.$$

Finally (20) and (21) may be written

$$(22)\quad d_{n,s}=\prod_{j=1}^{\nu}\prod_{h=1}^{n-1}\Big[E+(s+h\lambda+j-1)(s+[n-h]\lambda+2\nu-j)\Big]\cdot d_{1,s},$$

which is the formula sought.

5. *Explicit expressions.* If $\Omega(n)$ denote the operator

$$A_0(s,n)E^N+\cdots+A_N(s,n),$$

then evidently, from

$$a_{n+1,s}=\Omega(n)a_{n,s},$$

we get

$$(23)\qquad a_{n+1,s}=\Omega(n)\Omega(n-1)\cdots\Omega(1)a_{1,s};$$

hence if the product of operators on the right be expanded into a polynomial in E, the coefficients thus obtained will furnish the value of $a_{n+1,s}$. We shall limit ourselves in the following to the case in which $\Omega(n)$ can be split into a product of permutable linear factors. It will be noticed that all the special cases treated above actually lead to operators $\Omega(n)$ of this type.

We first investigate the conditions under which an operator

$$\Omega=E^k+\cdots+A_k(s)$$

may have this property. If we define $B_\nu(s)$ by the equation

(24) $$A_\nu(s) - A_\nu(s-1) = A_{\nu-1}(s)B_{\nu-1}(s) \qquad (\nu = 1, \cdots, k),$$

(where $A_0(s) \equiv 1$) then we shall prove that *a necessary and sufficient condition that Ω be a product of permutable linear operators is furnished by*

(25) $$B_{k-\nu}(s) = B_{k-1}(s) + B_{k-1}(s-1) + \cdots + B_{k-1}(s-\nu+1) \qquad (\nu = 2, 3, \cdots, k).$$

To prove the *necessity*, we remark first that two permutable linear operators must be of the form

$$E + \alpha(s), \qquad E + \alpha(s) + \mu,$$

where μ is free of s. If then

$$\Omega = \prod_{\nu=1}^{k} (E + \alpha(s) + \mu_\nu),$$

it is clear that

$$(E + \alpha(s))\Omega = \Omega(E + \alpha(s)),$$

and therefore

$$\Omega E - E\Omega = \alpha(s)\Omega - \Omega\alpha(s).$$

But

(26) $$\Omega E - E\Omega = \sum (A_{k-\nu}(s) - A_{k-\nu}(s-1))E^{\nu+1},$$

and

$$\alpha(s)\Omega - \Omega\alpha(s) = \sum A_{k-\nu}(s)[\alpha(s) - \alpha(s-\nu)]\, E^\nu.$$

Comparing coefficients we find that

$$A_{k-\nu+1}(s) - A_{k-\nu+1}(s-1) = A_{k-\nu}(s)[\alpha(s) - \alpha(s-\nu)];$$

therefore, by (24),

$$B_\nu(s) = \alpha(s) - \alpha(s-k+\nu),$$

and (25) follows immediately.

To prove the *sufficiency*, write

$$\beta(s) = \sum_{\nu=1}^{s} B_{k-1}(\nu),$$

so that, using (25),

$$B_{k-\nu}(s) = \beta(s) - \beta(s-\nu) \qquad (\nu = 1, \cdots, k).$$

But writing equation (26) in the form

$$\Omega E - E\Omega = \sum A_{k-\nu}(s)B_{k-\nu}(s)E^\nu,$$

it is immediately apparent that

(27) $$(E + \beta(s))\Omega = \Omega(E + \beta(s)).$$

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In order to complete the proof, we have then to prove the

LEMMA. *If an operator Ω satisfy equation (27), then it may be written as a product of permutable linear factors.*

If we put

$$\Omega_1 = \Omega - (E + \beta(s))^k,$$

it is evident that

$$(E + \beta(s))\Omega_1 = \Omega_1(E + \beta(s)),$$

so that the coefficient of E^{k-1} in Ω_1 is free of s; call it p_1. We next put

$$\Omega_2 = \Omega_1 - p_1(E + \beta(s))^{k-1},$$

and it appears in exactly the same way that p_2, the coefficient of E^{k-2} in Ω_2, is free of s. Continuing in this way we see that we may write

$$\Omega = (E + \beta(s))^k + p_1(E + \beta(s))^{k-1} + \cdots + p_k, \tag{28}$$

where the p_ν are all free of s. Therefore, finally if $\rho_1, \cdots, \rho_k$ are the roots of the equation

$$\rho^k + p_1\rho^{k-1} + \cdots + p_k = 0,$$

we have

$$\Omega = (E + \beta(s) + \rho_1) \cdots (E + \beta(s) + \rho_k),$$

completing the proof.

Returning to equation (23), we consider the expansion into a polynomial of

$$\Omega = \prod_{\nu=1}^{k} (E + \alpha_\nu(s, n)) = \sum_{i=0}^{k} A_i^{(k)}(s, n)E^i, \tag{29}$$

where

$$\alpha_\nu(s, n) = \alpha(s) + \rho_\nu(n). \tag{30}$$

We take first the case in which

$$\rho_\nu(n) \equiv 0 \qquad (\nu = 1, \cdots, k).$$

It is then fairly clear that $A_i^{(k)}$ may be written

$$A_i^{(k)}(s, n) = A_i^k(s) = \sum_{j=0}^{i} B_{ij}(s)\alpha^k(s - j). \tag{31}$$

To solve for B_{ij}, note that

$$A_i^i = 1, \quad A_i^k = 0 \quad \text{for} \quad k < i.$$

Then by Cramer's Rule, (31) yields

$$B_{ij}(s) = \frac{(-1)^j}{\prod_{\mu=0}^{j-1} [\alpha(s - \mu) - \alpha(s - j)] \prod_{\nu=j+1}^{i} [\alpha(s - j) - \alpha(s - \nu)]}, \tag{32}$$

after some easy transformations.

To treat the general case of $A_i^{(k)}(s,n)$ in equation (29), we note that, exactly as in (28), for the Ω now under discussion,

$$\Omega \equiv (E+\beta(s))^k + p_1(n)(E+\beta(s))^{k-1} + \cdots + p_k(n),$$

where $p_\nu(n)$ is the ν-th elementary symmetric function of the quantities $\rho_1(n), \cdots, \rho_k(n)$. Therefore, by (29) and (31), we find finally that

$$A_i^{(k)}(s,n) = \sum_{j=0}^{i} B_{ij}(s)\alpha_1(s-j,n)\cdots\alpha_k(s-j,n). \tag{33}$$

Equation (33), together with (32), furnishes the desired explicit expression for the coefficients in (29).

Hence, for the array

$$a^1_{n+1,s} = [E+\alpha(s)+\beta(n)]\cdot a^1_{n,s},$$

from

$$a^1_{n+1,s} = [E+\alpha(s)+\beta(n)]\cdots[E+\alpha(s)+\beta(1)]\cdot a^1_{1,s},$$

we derive immediately

$$a^1_{n,s} = \sum_{j=0}^{s-1} \frac{(-1)^j[\alpha(s-j)+\beta(1)]\cdots[\alpha(s-j)+\beta(n)]}{\prod_{h=0}^{j-1}[\alpha(s-h)-\alpha(s-j)]\prod_{t=j+1}^{s-1}[\alpha(s-j)-\alpha(s-t)]}.$$

For the array

$$a^\mu_{n+1,s} = [E+\alpha(s)+\beta_1(n)]\cdots[E+\alpha(s)+\beta_\mu(n)]\cdot a^\mu_{n,s}$$

we consider the operator

$$\prod_{i=1}^{\mu}\prod_{j=1}^{n}[E+\alpha(s)+\beta_i(j)].$$

Accordingly in (30) take

$$k = n\mu; \qquad \rho_1(n)=\beta_1(1), \cdots, \rho_k(n)=\beta_\mu(n).$$

Then

$$a^\mu_{n+1,s} = \sum_{j=0}^{s-1} \frac{(-1)^j\prod\prod[\alpha(s-j)+\beta_h(t)]}{\prod[\alpha(s-h)-\alpha(s-j)]\prod[\alpha(s-j)-\alpha(s-t)]}$$

$$= \sum_{j=1}^{s} \frac{(-1)^{s-j}\prod_{h=1}^{\mu}\prod_{t=1}^{n}[\alpha(j)+\beta_h(t)]}{\prod_{h=j+1}^{s}[\alpha(h)-\alpha(j)]\prod_{t=1}^{j-1}[\alpha(j)-\alpha(t)]}, \tag{34}$$

which furnishes the desired explicit expression.

6. *Explicit expressions for the arrays of* §§ 1, 3. By means of (32) or

(34) we can immediately write down simple expressions for the elements of the arrays defined in §§ 1, 3. For (5), (9),

$$a_{n+1,s}=\prod_{h=1}^{\mu}[E+s+n\lambda+h-1]\cdot a_{n,s};$$

take

$$\alpha(s)=s,\quad \beta_h(t)=t\lambda+h-1;$$

then

$$(35)\quad a_{n+1,s}=\frac{1}{(s-1)!}$$
$$\times\sum_{j=1}^{s}(-1)^{s-j}\binom{s-1}{j-1}(j+\lambda+\mu-1)!^{\mu}\cdots(j+n\lambda+\mu-1)!^{\mu};$$

the right side may be transformed into a sum depending on λ and μ, but it is unnecessary to consider that here.

Turning now to (15), we take in (30), (32), (33)

$$\alpha(s)=s^2+([2n-1]\lambda+\mu-1)s;$$
$$\rho_{(j-1)\mu+h}(n)=(j\lambda+h-1)([2n-j-1]\lambda+\mu-h)$$
$$(j=1,\cdots,n-1;\quad h=1,\cdots,\mu);$$

then by (34) after some easy reduction,

$$(36)\quad c_{n,s}=\frac{1}{(2s+[2n-1]\lambda+\mu-1)!}\sum_{j=1}^{s}(-1)^{s-j}\binom{2s+[2n-1]\lambda+\mu-1}{s-j}$$
$$\cdot(2j+[2n-1]\lambda+\mu-1)(j+[2n-1]\lambda-1)!^{[2n-1]\lambda}\prod_{h=1}^{2n-1}(j+h\lambda+\mu-1)!^{\mu}.$$

To determine the $d_{n,s}$ of (16) we make use of (22), and take

$$\alpha(s)=s^2+(n\lambda+2\nu-1)s;$$
$$\rho_{(j-1)\nu+h}(n)=(j\lambda+h-1)([n-j]\lambda+2\nu-h)$$
$$(j=1,\cdots,n-1;\quad h=1,\cdots,\nu).$$

Then exactly as in deriving (32),

$$(37)\quad d_{n,s}=\frac{1}{(2s+n\lambda+2\nu-1)!}\sum_{j=1}^{s}(-1)^{s-j}\binom{2s+n\lambda+2\nu-1}{s-j}$$
$$\cdot(2j+n\lambda+2\nu-1)(j+n\lambda-1)!^{n\lambda}\prod_{h=1}^{n}(j+h\lambda+2\nu-1)!^{2\nu}.$$

7. *Some applications.* If we operate on x^m with both members of (6) we find that

$$m!^{\lambda+\mu}\cdots(m-[n-1]\lambda)!^{\lambda+\mu}=\sum_{s=1}^{\mu(n-1)+1} b_{n,s}m!^{s+\mu-1+n\lambda}.$$

Employing the identity

$$\sum_{m=1}^{m} m!^{n}=\frac{(m+1)!^{n+1}}{n+1}$$

we get the summation

$$(38)\quad \sum_{m=1}^{m} m!^{\lambda+\mu}\cdots(m-[n-1]\lambda)!^{\lambda+\mu}=\sum_{s=1}^{\mu(n-1)+1} b_{n,s}\frac{(m+1)!^{s+\mu+n\lambda}}{s+\mu+n\lambda};$$

the right member may be written as an explicit function of m, n, λ, μ by substituting from (35).

Better summations for the left member of (38) are obtained when either μ is even or n is odd. Thus by means of (11),

$$(39)\quad \sum_{m=1}^{m} m!^{\lambda+\mu}\cdots(m-2(n-1)\lambda)!^{\lambda+\mu}$$
$$=\sum_{s=1}^{\mu(n-1)+1} c_{n,s}\frac{(m+s)!^{[2n-1]\lambda+2s+\mu-1}}{(2n-1)\lambda+2s+\mu-1};$$

while from (16),

$$(40)\quad \sum_{m=1}^{m} m!^{\lambda+2\nu}\cdots(m-[n-1]\lambda)!^{\lambda+2\nu}$$
$$=\sum_{s=1}^{\nu(n-1)+1} d_{n,s}\frac{(m+s)!^{n\lambda+2s+2\nu-1}}{n\lambda+2s+2\nu-1};$$

the right members contain approximately but half the number of terms in the right member of (38). Of course (38), (39), (40) can only be spoken of as summations—in a practical sense—when m is large as compared with $n\mu$ (or $n\nu$).

There is a curious connection between (38) and certain Bernoulli polynomials in several variables that I shall consider in detail elsewhere. Limiting ourselves here to the case $\lambda=0$, and modeling after the well known expression of the Bernoulli numbers in terms of the Stirling numbers,* we define $(n_i \geqq 0)$

$$B_n(\xi)=B_{n_1\ldots n_\mu}(\xi_1\cdots\xi_\mu)=\sum_{s=0}^{\infty}\frac{1}{s+1}\sum_{\alpha=0}^{s}(-1)^{\alpha}\binom{s}{\alpha}(\alpha+\xi_1)^{n_1}\cdots(\alpha+\xi_\mu)^{n_\mu}.$$

Note that the outer sum is actually finite and that the inner sum is a generalization of (35) $(\lambda=0)$. Then

* J. Worpitzky, "Studien über die Bernoullischen und Eulerschen Zahlen," *Journal für Mathematik*, Vol. 94 (1883), p. 215.

$$\sum_n B_n(\xi)\frac{x_1^{n_1}\cdots x_\mu^{n_\mu}}{n_1!\cdots n_\mu!}=\sum_{s=0}^{\infty}\frac{1}{s+1}\sum_{a=0}^{s}(-1)^a\binom{\ }{\ }e^{(a+\xi_1)x_1+\cdots+(a+\xi_\mu)x_\mu}$$
$$=\sum_{s=0}^{\infty}\frac{1}{s+1}(1-e^{x_1+\cdots+x_\mu})^s\,e^{\xi_1x_1+\cdots+\xi_\mu x_\mu}$$
$$=\frac{(x_1+\cdots+x_\mu)\,e^{\xi_1x_1+\cdots+\xi_\mu x_\mu}}{e^{x_1+\cdots+x_\mu}-1},$$

and this last furnishes a convenient definition for the $B_n(\xi)$. Now on the other hand let

$$S\binom{n}{\xi}=S\begin{pmatrix}n_1\cdots n_\mu\\ \xi_1\cdots\xi_\mu\end{pmatrix}=\sum_{m=0}^{m-1}(m+\xi_1)^{n_1}\cdots(m+\xi_\mu)^{n_\mu}.$$

Evidently then

$$\sum_n S\binom{n}{\xi}\frac{x_1^{n_1}\cdots x_\mu^{n_\mu}}{n_1!\cdots n_\mu!}=\sum_{m=0}^{m-1}e^{(m+\xi_1)x_1+\cdots+(m+\xi_\mu)x_\mu}$$
$$=\frac{e^{m(x_1+\cdots+x_\mu)}-1}{x_1+\cdots+x_\mu}\,\frac{(x_1+\cdots+x_\mu)\,e^{\xi_1x_1+\cdots+\xi_\mu x_\mu}}{e^{x_1+\cdots+x_\mu}-1}$$
$$=\sum_{r=1}^{\infty}\frac{(x_1+\cdots+x_\mu)^{r-1}m^r}{r!}\sum_n B_n(\xi)\frac{x_1^{n_1}\cdots x_\mu^{n_\mu}}{n_1!\cdots n_\mu!}.$$

Equating coefficients,

$$S\binom{n}{\xi}=\sum_{t_i=0}^{n_i}\binom{n_1}{t_1}\cdots\binom{n_\mu}{t_\mu}B_{n-t}(\xi)\frac{m^{t_1+\cdots+t_\mu+1}}{t_1+\cdots+t_\mu+1}.^*$$

Again if we put

$$T\binom{n}{\xi}=T\begin{pmatrix}n_1\cdots n_\mu\\ \xi_1\cdots\xi_\mu\end{pmatrix}=\sum_{m=0}^{m-1}(-1)^m(m+\xi_1)^{n_1}\cdots(m+\xi_\mu)^{n_\mu},$$

and define the Euler polynomials $E_n(\xi)$ by

$$\frac{2e^{\xi_1x_1+\cdots+\xi_\mu x_\mu}}{e^{x_1+\cdots+x_\mu}+1}=\sum_n E_n(\xi)\frac{x_1^{n_1}\cdots x_\mu^{n_\mu}}{n_1!\cdots n_\mu!},$$

it is easily seen that

$$T\binom{n}{\xi}=\tfrac{1}{2}E_n(\xi)-\frac{(-1)^m}{2}E_n(m+\xi).$$

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* For the function

$$S'\binom{\eta}{\xi}=\sum_{t=0}^{m-1}(d/dt)\,(t+\xi_1)^{n_1}\cdots(t+\xi_\mu)^{n_\mu}$$

there is the much simpler formula

$$S'\binom{\eta}{\xi}=B_n(\xi+m)-B_n(\xi).$$

A CLASS OF DYNAMICAL SYSTEMS ON SURFACES OF REVOLUTION.

By G. Baley Price.

Introduction. A heavy particle on a surface of revolution furnishes the simplest example of a general class of dynamical systems on surfaces of revolution in which the force function depends only on the latitude and not on the longitude. The problem of the heavy particle and other special cases obtained by restricting both the force function and the surface have been treated by Jacobi † and other writers.‡ But even these treatments of special cases of the problem are not in the spirit of modern dynamics and do not use modern methods. The present paper treats the most general case of the problem; the methods used are those of surfaces of section and surface transformations which have been developed by Poincaré and Birkhoff. The problem furnishes a simple example of an integrable dynamical system with two degrees of freedom.

In this exposition only surfaces of genus one are considered, but it is clear that the same methods apply with essentially the same results in the case of surfaces of genus zero. Also, the present problem includes as a special case the determination of the geodesics on the surface.§

1. *The surface and the force function.* In a space with rectangular coördinates (ξ, η, ζ) we shall consider a surface of revolution S which has the ζ-axis as axis of revolution. The surface is generated by rotating about the ζ-axis a simple closed curve whose parametric equations in the (r, ζ) plane of rectangular coördinates are

$$r = r(x), \qquad \zeta = \zeta(x), \tag{1}$$

where $r(x)$, $\zeta(x)$ are functions satisfying the following hypotheses:

$$r(x + \omega) = r(x), \qquad \zeta(x + \omega) = \zeta(x), \tag{2}$$

† Jacobi, *Journal für Mathematik*, Vol. 24 (1842), pp. 5-27.

‡ Gustaf Kobb, "Sur le mouvement d'un point matériel sur une surface de révolution," *Acta Mathematica*, Vol. 10 (1887), pp. 89-108; Otto Staude, "Über die Bewegung eines schweren Punktes auf einer Rotationsfläche," *Acta Mathematica*, Vol. 11 (1888), pp. 303-332.

§ B. F. Kimball, "Geodesics on a Toroid," *American Journal of Mathematics*, Vol. 52 (1930), pp. 29-52; G. A. Bliss, "The Geodesic Lines on the Anchor Ring," *Annals of Mathematics*, 2nd ser., Vol. 4 (1902), pp. 1-21.

(3) $r(x),\ \zeta(x)$ are analytic, $\quad 0 \leqq x \leqq \omega,$

(4) $r(x) \geqq k > 0, \quad k$ a constant.

We assume also that x is the arc length on the curve (1), i. e.,

(5) $$r_x{}^2 + \zeta_x{}^2 = 1.$$

A subscript x here denotes, as throughout the paper, a derivative with respect to x.

If y is the angle which the plane through the ζ-axis and the point (ξ, η, ζ) forms with the (ξ, ζ) plane, then the parametric equations of S are

(6) $$\xi = r(x) \cos y, \qquad \eta = r(x) \sin y, \qquad \zeta = \zeta(x).$$

The curves $x =$ constant and $y =$ constant on S will be called *parallels* and *meridians* respectively. It is clear that S is a surface of genus one.

On S we shall consider the motion of a particle of mass m under the action of forces derived from the force function $U = mu(x)$, where $u(x)$ satisfies the following hypotheses:

(7) $u(x + \omega) = u(x),$

(8) $u(x)$ is analytic, $\quad 0 \leqq x \leqq \omega.$

2. *The equations of motion.* Because of (5) we find that the kinetic energy T is given by $T = (m/2)(x'^2 + r^2y'^2)$. Here, as throughout the paper, primes denote derivatives with respect to the time t. The two equations of motion in the Lagrangian form give

(9) $$r^2y' = c, \qquad \text{[the integral of areas]}$$

where c is a constant of integration, and

(10) $$x'' = rr_xy'^2 + u_x.$$

The integral of energy is

(11) $$x'^2 + r^2y'^2 = 2(u + h).$$

We may write (9) and (10) in the form of a first order system of differential equations. In this form all of the usual existence theorems can be applied [R 1, pp. 1-14] (we refer in this fashion to the references at the end).

We may use (9) to eliminate y' from (10) and (11). We obtain

(12) $$x'' = (c^2r_x + r^3u_x)/r^3,$$

(13) $$x'^2 = [2r^2(u + h) - c^2]/r^2.$$

For later use we write these equations in the following notation. Define two functions v and w as follows:

$$v = 2r^2(u + h), \tag{14}$$

$$w = -r^3 u_x / r_x, \qquad r_x \neq 0. \tag{15}$$

Then equations (12) and (13) become

$$x'' = r_x(c^2 - w)/r^3, \qquad r_x \neq 0 \tag{16}$$

$$x'^2 = (v - c^2)/r^2. \tag{17}$$

There are two further formulas which we shall need later. Let $t(x^0x)$ and $y(x^0x)$ designate the time required for the particle to pass from $x = x^0$ to $x = x$ along a trajectory and the angle y through which it moves respectively. Then from (9) and (17) we obtain the following formulas:

$$\begin{aligned} t(x^0x) &= \pm \int_{x^0}^{x} \frac{r\,dx}{(v - c^2)^{1/2}}. \\ y(x^0x) &= \pm \int_{x^0}^{x} \frac{c\,dx}{r(v - c^2)^{1/2}}, \end{aligned} \tag{18}$$

3. *A first surface of section.*

A. The manifold of states of motion and the surface of section. The *manifold of states of motion* [see R 2, Part III for the methods of this section] is the three dimensional manifold in four space obtained by specifying the value of the energy constant h in the integral of energy (11). The integral of areas gives a second integral. Since two integrals exist, the system is said to be *integrable.* The trajectories lie on certain invariant sub-manifolds of (11) which are obtained by specifying the value of c in (9).

According to Birkhoff a *surface of section* is defined as an analytic surface in the manifold of states of motion, "regularly bounded" by a finite number of closed stream lines, cut throughout in the same sense by the stream lines and at least once by every stream line in a fixed interval of time. In this section we propose to obtain a surface of section and show how it can be used to study the trajectories of the system.

In the present section we shall suppose that h has a value such that motion takes place over only a part of S; in section 4 we shall treat the case in which motion takes place over the entire surface. We now make the assumption about the region of motion more precise as follows. Since v is not a constant, we find that it is possible to choose h so that there are regions of motion in which v_x vanishes on only a single parallel. There may be more than one such region for a given value of h, but we consider just one of them. Thus we assume that h is fixed and so chosen that v_x vanishes on only a single parallel, $x = x^*$, in the region of motion under consideration. Then v has a maximum at $x = x^*$, at which we shall assume $v_{xx} < 0$. The motion

on S takes place in the region $v \geqq 0$; this region is bounded by the two parallels on which $v = 0$. Then it is easily found that the parallel $x = x^*$ is a closed periodic trajectory.

We proceed to give a representation of the manifold of states of motion in 3-space. Set

$$\tan \psi = ry'/x', \qquad -\pi \leqq \psi \leqq \pi \tag{19}$$

and let ψ vary continuously over the indicated interval as the velocity vector (x', y') turns about (x, y); furthermore, $x' > 0$, $y' = 0$ shall correspond to $\psi = 0$, and the sign of ψ shall be the same as that of y'. Since S is a surface of revolution, ψ is an angle which the trajectory makes at each point with the meridian. To each point (x, y, x', y') in the manifold of states of motion, there corresponds a point (x, y, ψ) in the representation in 3-space; the points (x, y, π) and $(x, y, -\pi)$ are to be considered identical since they correspond to the same point in the original manifold. This representation is valid except at points where $x' = y' = 0$, i. e., except at points on an oval of zero velocity. The representation is certainly valid in the neighborhood of $x = x^*$.

We shall now show that the ring surface

$$x = x^*, \qquad -\pi/2 \leqq \psi \leqq \pi/2, \qquad 0 \leqq y < 2\pi$$

forms a surface of section. In the first place, this surface has two boundaries which correspond to a periodic trajectory traced in the two possible directions. In the second place, we find from the equations of motion that the trajectories oscillate between two parallels on S. Every trajectory therefore cuts through the surface in one and the same sense, and (18) shows that the length of time between any two successive crossings is finite except possibly in the neighborhood of the boundaries of the surface. To determine what happens near the boundaries, we have recourse to the equation of normal displacement for $x = x^*$. If $x = x^* + \epsilon\bar{x}$, $y = y^* + \epsilon\bar{y}$ is a nearby trajectory, a calculation shows that in the limit $\bar{x}$ satisfies the equation

$$\frac{d^2\bar{x}}{dt^2} = \left(\frac{v_{xx}}{2r^2}\right)_{x^*} \bar{x}. \tag{20}$$

Hence, since $v_{xx} < 0$ by hypothesis, the limiting length of time between successive crossings as the point of crossing approaches the boundaries of the surface is $2\pi r(2)^{1/2}/(-v_{xx})^{1/2}$. In the third place, no trajectory is tangent to the surface, for a trajectory pierces the surface with the direction components (x', y', ψ'), and x' vanishes only on the boundaries $\psi = \pm\pi/2$. It can be proved easily that the angle at which a trajectory pierces the surface is of the first order in the distance to the boundary. We are thus justified in

calling $x = x^*$, $-\pi/2 \leqq \psi \leqq \pi/2$ a surface of section, because it satisfies the three requirements of the definition.

B. The transformation T. The transformation T is defined as follows: a trajectory which pierces the surface of section at P has its next succeeding intersection at P'. Then $P' = T(P)$.

Now from (19) and (11) we find $ry' = [2(u+h)]^{1/2} \sin\psi$. Then from (9) we have $v^{1/2} \sin\psi = c$. But c is constant along a given trajectory; hence, T has the invariant function $v^{1/2} \sin\psi$. Since x is constant on the surface of section, it follows that *the circles $\psi =$ constant are path curves of T*. Furthermore, it is clear from (18) that $y(x^0 x)$ is independent of y; hence, T has the form

$$\bar{\psi} = \psi, \qquad \bar{y} = y + \alpha,$$

where α depends on ψ but not on y. Thus T transforms the ring-shaped surface of section into itself by rotating each circle $\psi =$ constant into itself. The amount of rotation α on each circle is called the *rotation number* [R 3, pp. 87-88] of T on that circle. The trajectories oscillate between two parallels on S, and α is the increase in y in one complete oscillation. From (18) we find

$$\alpha = 2 \int_{x_1}^{x_2} \frac{c\,dx}{r(v - c^2)^{1/2}}$$

where x_1 and x_2 are the minimum and maximum values of x respectively on the trajectories for the given value of c. The rotation on the boundaries is found from (20) and (9) to be $2\pi c(2)^{1/2}/r(-v_{xx})^{1/2}$. Thus $\alpha > 0$ when $c > 0$, i. e., when $\psi > 0$, and $\alpha < 0$ when $\psi < 0$. On $\psi = 0$ we have $\alpha = 0$. Also, α is a continuous function of c.

If $\alpha = 2q\pi/p$, where p and q are integers without common factors, on a circle $\psi =$ constant, this circle is rotated into its original position by T^p. Then the circle represents a family of closed, periodic trajectories which we designate as of type (p, q). Here $p > 0$ represents the number of complete oscillations between two parallels which the trajectory makes before closing, and q, positive or negative, is the number of multiples of 2π by which y increases.

Let α_M be the maximum rotation number on the entire surface of section and α_B the rotation number on $\psi = \pi/2$. Then since α varies continuously on the circles $\psi =$ constant, we see at once that the following theorem is true.

THEOREM. *If $\alpha_B < \alpha_M$, there are at least two families of closed, periodic trajectories of each type (p, q), where $\alpha_B < |2q\pi/p| < \alpha_M$; in any case,*

there is at least one family of closed, periodic trajectories of each type (p, q) *where* $0 \leqq |2q\pi/p| < \alpha_B$.

We state without going into details at this point that the circles on which α is an irrational multiple of 2π represent families of recurrent trajectories.

It is possible to show that there is an invariant volume integral in the manifold of states of motion, and that T has an invariant area integral on the surface of section. The surface transformation here considered is then a degenerate case of Poincaré's Last Geometric Theorem [R 4; R 2, p. 294]. It illustrates also one of the three general types of fixed points in surface transformations [R 3, p. 4, type III]. The surface of section and transformation could be extended to some extent to other cases of the problem, but we prefer to consider now a second type of surface of section, which is more interesting in the general case.

4. *A second surface of section.*

A. The manifold of states of motion. On the surface S consider a general dynamical system in which the force function is $U = mu(x, y)$, where $u(x, y)$ is an arbitrary analytic function which is periodic with periods ω and 2π in x and y respectively. Then the equations of motion are

$$(21) \qquad x'' = u_x + rr_x y'^2, \qquad y'' = (u_y - 2rr_x x'y')/r^2,$$

and the integral of energy is

$$(22) \qquad x'^2 + r^2 y'^2 = 2(u + h).$$

According to the definition given in section 3, the equation of the manifold of states of motion is (22). We can give a representation of this manifold in 3-space as follows. Set

$$(23) \qquad \tan\phi = x'/ry', \qquad -\pi \leqq \phi \leqq \pi$$

and let ϕ vary continuously over the interval indicated as the vector (x', y') turns about (x, y). Also, let $\phi = 0$ when $x' = 0$, $y' > 0$, and let ϕ have the same sign as x'. Since S is a surface of revolution, ϕ is an angle which the trajectory makes with the parallel at each point. Corresponding to each state of motion (x, y, x', y') there is a point (x, y, ϕ) in the representation; the points (x, y, π) and $(x, y, -\pi)$ are to be considered identical. The totality of points (x, y, ϕ) form a manifold M in 3-space. This representation is valid except at points where $x' = y' = 0$.

Now the totality of states of motion along a trajectory corresponds to a curve, a *stream line,* in M. The differential equations of these stream lines can be found as follows. From (23), $x' = \rho\sin\phi$, $ry' = \rho\cos\phi$. Substitute

in (22) to determine ρ. A straightforward calculation gives ϕ'. The results are

$$(24)\qquad \begin{cases} x' = X(x, y, \phi) \equiv [2(u+h)]^{1/2} \sin\phi \\ y' = Y(x, y, \phi) \equiv (1/r)[2(u+h)]^{1/2} \cos\phi \\ \phi' = \Phi(x, y, \phi) \equiv \dfrac{ru_x \cos\phi - u_y \sin\phi + 2r_x(u+h)\cos\phi}{[2r^2(u+h)]^{1/2}} \end{cases}$$

Now the time t can be eliminated completely from (24); we obtain

$$(25)\qquad dx/X = dy/Y = d\phi/\Phi.$$

The trajectories thus appear in M as the stream lines of the steady fluid motion defined by (25).

A fundamental property of the steady flow in M is stated in the following lemma.

LEMMA 1. *The volume integral*

$$(26)\qquad \iiint r(x)\, dx\, dy\, d\phi$$

is invariant in the steady flow in M defined by (25).

A necessary and sufficient condition that (26) be invariant is that [R 5, pp. 285-286; R 6, vol. III, pp. 1-6]

$$\frac{\partial(rX)}{\partial x} + \frac{\partial(rY)}{\partial y} + \frac{\partial(r\Phi)}{\partial\phi} \equiv 0$$

and we find by substituting from (24) that this condition is satisfied.

We may think of $r(x)$ as the density function in M. Then (26) represents the mass.

B. The surface of section. We return now to a consideration of the special systems treated in this paper, i. e., systems in which $u = u(x)$. We shall assume that h is so chosen that $v > 0$ for all values of x. Then the meridian $y = 0$ is a closed periodic trajectory, and we shall show how a surface of section of a certain type can be constructed from it. In the manifold M, a ring-shaped surface R is defined by $y = 0$, $y' \geqq 0$.

Now in the first place, the boundaries of R are $y = 0$, $y' = 0$ with either $x' > 0$ or $x' < 0$. They are therefore two closed stream lines which correspond to a closed trajectory traced in the two possible directions. In the second place, no stream line is tangent to R, and the angle at which a stream line cuts R is of the first order in the distance to the boundary, for if θ is the angle,

$$\sin\theta = y'/(x'^2 + y'^2 + \phi'^2)^{1/2}.$$

Now the distance to the boundaries is $| \pm \pi/2 - \phi |$. Then

$$\lim_{\phi \to \pm\pi/2} \frac{\theta}{| \pm \pi/2 - \phi |} = \lim_{\phi \to \pm\pi/2} \frac{\sin\theta}{| \pm \pi/2 - \phi |}.$$

Substitute now for $\sin\theta$, and then substitute from the equations corresponding to (24). We find that the limit is $1/r \neq 0$, from which the stated results follow.

In the third place, we must consider the intersections of the trajectories with R. Now we may consider $y' \geqq 0$ on every trajectory, because negative values of y' merely give the same trajectories traced in the opposite direction. The trajectory $y = 0$, $y' = 0$ forms the boundaries of R, but no other trajectory with $y' = 0$ has any point in common with R. Therefore R does not satisfy the requirement for a surface of section that *all* trajectories intersect it. Now (9) shows that all trajectories which are not meridians have $y' > 0$ always; hence, sooner or later they cross R. However, as c approaches zero, the maximum value of y' approaches zero; hence, the interval of time between successive crossings of R becomes infinite. The intersections with R approach the boundaries as c approaches zero. In any closed region $r^2y' \geqq c^0 > 0$ inside the ring R the interval of time between successive crossings is uniformly bounded. We see therefore that R fails to satisfy a second requirement for a surface of section.

LEMMA 2. *In the sense explained, the surface $R: y = 0$, $y' \geqq 0$ forms a surface of section.*

C. The transformation T. The transformation T on R is defined in the usual way: a stream line which crosses R at P has its first succeeding crossing at P'. Then $P' = T(P)$. This transformation carries R into itself in a one-to-one and continuous manner and is analytic in the interior of the ring. Results to be established presently make it clear that T cannot be analytic along the boundaries of R.

A fundamental property of T is established in the following lemma.

LEMMA 3. *The transformation T on R has the positive invariant area integral*

$$\iint [2(u+h)]^{1/2} \cos\phi \, dx \, d\phi, \tag{27}$$

and its value over the entire ring is finite.

Consider a region σ on R and the region σ' into which it is transformed by T. By lemma 1, the mass of the tube of stream lines bounded by σ and σ' is invariant as it moves along. The rate of decrease of mass at the base σ is [R 5, pp. 286-287]

$$\int\int_{\sigma} ry'\,dx\,d\phi,$$

and the rate of increase at the base σ' is given by the same integral extended over the region σ'. It follows that these two integrals are equal; hence, substituting for y' from (24) we find that (27) is an invariant integral. The value of the integral extended over the entire ring is $2\int_0^{\omega}[2(u+h)]^{1/2}\,dx$, and this is positive and finite.

If we consider $[2(u+h)]^{1/2}\cos\phi$ the density of R, then (27) is the mass.

The following lemma gives a second important property of T.

LEMMA 4. *The function $F \equiv v\cos^2\phi$ is invariant under T.*

From (9) and the equations corresponding to (24) we find that $F = c^2$. But since c is constant along a given trajectory, the lemma follows.

The existence of F is a direct consequence of the fact that the system is integrable, i. e., that two integrals (9) and (11) are known. Now F is positive over the interior of R and vanishes on the boundaries. Furthermore, it is symmetric in $\phi = 0$, and for a fixed value of x decreases monotonically as $|\phi|$ approaches $\pi/2$. It follows that the critical points of F, i. e., points (x, ϕ) at which the two first partial derivatives of F vanish [Morse's definition], which are interior points of R lie on the line $\phi = 0$. The level curves of F are path curves of T.

D. The path curves of T. As we have just seen, the value of c determines the nature of the path curve $F = c^2$ which corresponds to the trajectories for the given value of c. First we shall determine the types of path curves, and then study the nature of the corresponding trajectories.

We shall suppose that R is taken as a ring bounded by two concentric circles. The coördinates on R are (x, ϕ), where x varies from 0 to ω around the ring in the counter clockwise direction, and ϕ varies from $-\pi/2$ on the inner boundary to $\pi/2$ on the outer boundary. The path curves $F = c^2$ are symmetric in the circle $\phi = 0$.

Now plot $z = F$ over the ring R, and plot also $z = c^2$, a plane parallel to the ring. The projections on R of the curves of intersection of $z = F$ and the plane are the path curves. By letting c^2 vary from 0 to the maximum value of v, we obtain the totality of path curves.

In the first place, $z = F$ has a certain number of critical points, which are of special importance. For convenience we agree that *critical points of F* shall include only points on $\phi = 0$. Then the critical points of F are those

points and only those points on $\phi = 0$ for which $v_x = 0$. Now a calculation shows that v_x is given by

$$v_x = 2[2rr_x(u + h) + r^2 u_x], \tag{28}$$

$$v_x = 2r_x(v - w)/r, \qquad r_x \neq 0. \tag{29}$$

We now find from the equations of motion (12), (13) and (16), (17) that a necessary and sufficient condition that the parallel $x = x^*$ be a trajectory is that v_x vanish for $x = x^*$. Furthermore, there are two types of vanishing of v_x. From (28) the first occurs when $r_x = u_x = 0$, and from (29) the second when $v - w = 0$, $r_x \neq 0$. We may thus distinguish two types of critical points of $z = F$. The first of these two does not vary with h, but since w is independent of h and v depends on h, the second type of critical point does vary with h. Finally, since critical points of $z = F$ correspond to trajectories which are parallels, we see that *critical points of F are invariant points of T.*

Now for certain values of c^2, the plane $z = c^2$ passes through critical points of $z = F$ which are not maxima (critical points which correspond to minima or points of inflection of the function $z = v$). Let these values of c^2, in the order of increasing magnitude, be designated by $c_0^2, c_1^2, \cdots, c_k^2$. Then each of the path curves $F = c_i^2$, $(i = 0, 1, \cdots, k)$, passes through an invariant point of T. If $z = v$ has a minimum at the critical point, the path curve has two branches which pass through the invariant point. If $z = v$ has a point of inflection, the path curve has a cusp at the critical point. We designate by P a critical point of $z = F$, and by C_1 a path curve, exclusive of the critical point or points, of the set $F = c_i^2$, $(i = 0, 1, \cdots, k)$.

For $c^2 = 0$, the trajectories are meridians. The corresponding curves $F = 0$ on R are the two boundaries. Now $F = c^2$ gives two path curves, designated by C_2, when $0 < c^2 < c_0^2$, one of which lies in the region between $F = c_0^2$ and the outer boundary of R, and the other of which lies between the same curve and the inner boundary. They are simple closed curves which can be deformed through points of R into the two boundaries of R.

Finally, $F = c^2$ gives one or more simple closed curves when $c^2 > c_0^2$, $c^2 \neq c_i^2$, $(i = 0, 1, \cdots, k)$. We designate these curves by C_3; they fill out the regions between the curves C_1.

We have thus found that the system has five types of trajectories: parallel trajectories corresponding to critical points P; meridian trajectories corresponding to the boundaries of R; and three other types corresponding to the path curves C_1, C_2, C_3.

E. Further properties of T. In the regions of R occupied by the path curves C_2 and C_3, it is possible to use x and c as coördinates instead of x

and ϕ. The transformation to the new coördinates is accomplished by means of the relation $c = v^{1/2} \cos \phi$. We thus find that the invariant area integral (27) expressed in terms of the new coördinates (x, c) is

$$\iint \frac{c\, dx\, dc}{r(v - c^2)^{1/2}}. \tag{30}$$

Since (30) is invariant, it follows that the integral

$$\int \frac{c\, dx}{r(v - c^2)^{1/2}} \tag{31}$$

is invariant under T on the path curves $F = c^2$ [R 3, pp. 93-94]. This integral diverges of course if integrated along a path curve C_1 up to a critical point of F.

Consider the path curves C_2. First we replace x by a new coördinate τ as follows. Set

$$\frac{\tau}{2\pi} \int_0^\omega \frac{c\, dx}{r(v - c^2)^{1/2}} = \int_0^x \frac{c\, dx}{r(v - c^2)^{1/2}}. \tag{32}$$

Then as x varies from 0 to ω, τ varies from 0 to 2π. We may obviously think of (c, τ) as polar coördinates.

LEMMA 5. *In the coördinates (c, τ), T is the rotation $c' = c$, $\tau' = \tau + \alpha$ on the path curves C_2.*

Let $x_1 : \tau_1$ and $x_2 : \tau_2$ be two points on a curve C_2 which are transformed by T into $x'_1 : \tau'_1$ and $x'_2 : \tau'_2$. Then we find

$$\frac{\tau_2 - \tau_1}{2\pi} \int_0^\omega \frac{c\, dx}{r(v - c^2)^{1/2}} = \int_{x_1}^{x_2} \frac{c\, dx}{r(v - c^2)^{1/2}},$$

and a similar equation in which x_1, x_2, τ_1, τ_2 are primed. But since (31) is invariant under T, we have $\tau'_2 - \tau'_1 = \tau_2 - \tau_1$. Set $\tau'_1 - \tau_1 = \alpha$ and drop the subscript 2. Then $\tau' = \tau + \alpha$ and the lemma is proved.

The number α is the rotation number of T on the given path curve. We compute α as follows.

$$\frac{\alpha}{2\pi} \int_0^\omega \frac{c\, dx}{r(v - c^2)^{1/2}} = \int_{x_1}^{x'_1} \frac{c\, dx}{r(v - c^2)^{1/2}}.$$

Now from (18)

$$\pm \int_{x_1}^{x'_1} \frac{c\, dx}{r(v - c^2)^{1/2}} = 2\pi$$

the plus or minus sign being used according as the path curve lies in the region $\phi > 0$ or $\phi < 0$. Substituting from the second equation in the first, we find

$$\alpha = \pm 4\pi^2 / \int_0^\omega \frac{c\, dx}{r(v - c^2)^{1/2}}. \tag{33}$$

From this equation we see that on the path curves C_2 in the region $\phi > 0[\phi < 0] 0 < \alpha < +\infty[-\infty < \alpha < 0]$; as c decreases from c_0 to 0, α increases [decreases] monotonically and continuously from 0 to $+\infty$ [0 to $-\infty$]. This statement proves the following lemma.

LEMMA 6. *On the path curves C_2, α takes on every value except $\alpha = 0$ once and only once.*

Now consider the path curves C_3. We introduce a parameter τ on these curves as follows. Let x_1 and x_2 be the minimum and maximum values respectively of x on a given curve C_3. Set

$$\frac{\tau}{\pi}\int_{x_1}^{x_2}\frac{c\,dx}{r(v-c^2)^{1/2}} = \pm\int_{x_1}^{x}\frac{c\,dx}{r(v-c^2)^{1/2}}.$$

The integral on the right is to be integrated around the path curve in the counter clockwise direction, the plus sign being taken when $\phi > 0$ and the minus sign when $\phi < 0$. Then as the point with abscissa x traces the curve, τ increases from 0 to 2π. We may think of (c, τ) as polar coördinates on the curves C_3.

The proof of the following lemma is similar to that of lemma 5.

LEMMA 7. *In the coördinates (c, τ), T is the rotation $c' = c$, $\tau' = \tau + \alpha$ on the curves C_3.*

From (18) we find that on the curves C_3, α is given by

$$\alpha = 2\pi^2 \Big/ \int_{x_1}^{x_2}\frac{c\,dx}{r(v-c^2)^{1/2}}. \tag{34}$$

The rotation number on every path curve C_3 is therefore positive; α is a continuous function of c, and α approaches zero as c approaches c_i, $(i = 0, 1, \cdots, k)$.

The path curves C_3 fill out regions of two kinds: (a) ring shaped regions bounded on the inside and outside by a path curve C_1; (b) circular regions about an invariant point P, bounded on the outside by a curve C_1. Let α_M denote the maximum rotation number in these regions, and α_P the rotation number at a point P.

LEMMA 8. *On the curves C_3 in a ring region, α takes on each value on the interval $0 < \alpha < \alpha_M$ at least twice; on the curves C_3 in a circular region, α takes on each value $0 < \alpha < \alpha_P$ at least once, and if $\alpha_M > \alpha_P$, it takes on each value on the interval $\alpha_P < \alpha < \alpha_M$ at least twice.*

Now the curves C_1 are composed of arcs terminated by critical points of F. On each of these arcs introduce a new parameter τ as follows. Choose an interior point x^0 of the arc and set

$$\tau = \pm \int_{x^0}^{x} \frac{c\,dx}{r(v-c^2)^{1/2}},$$

where the plus (minus) sign is used when $\phi > 0$ ($\phi < 0$). Then τ increases from $-\infty$ to $+\infty$ as x increases (decreases) from one end of the arc to the other in $\phi > 0$ ($\phi < 0$). Let the points $x : \tau$ and $x^0 : 0$ be carried into $x' : \tau'$ and $x^* : h$ by T. Then since (31) is invariant under T, we find $\tau = \tau' - h$ or $\tau' = \tau + h$. From (18) we find

$$\int_{x^0}^{x^*} \frac{c\,dx}{r(v-c^2)^{1/2}} = 2\pi;$$

and hence T is the translation $\tau' = \tau + 2\pi$.

LEMMA 9. *In terms of the parameter τ, T on each arc of a curve C_1 is the translation $\tau' = \tau + 2\pi$; with reference to R, the motion is counter clockwise or clockwise according as $\phi > 0$ or $\phi < 0$.*

We have thus determined completely the nature of the transformation on R. We see that *the transformation T on the ring R is such that points are advanced in the region $\phi > 0$ and regressed in the region $\phi < 0$.* In paragraph D it was shown by using the equations of motion directly that the critical points of F are invariant points of T; this fact follows also from the properties of T which have been established in the present paragraph. We now see that T satisfies all the hypotheses of Poincaré's Last Geometric Theorem [R 4], for it is one-to-one and continuous in R; it has an invariant area integral; and it advances points in the neighborhood of the boundary $\phi = \pi/2$ and regresses points in the neighborhood of the boundary $\phi = -\pi/2$. It follows that T has at least two invariant points.

In the present problem it has been shown already that the invariant points of T are the critical points of F. At this point it is impossible to avoid the notion of multiple invariant points [R 2, p. 286]. It has been seen that each zero of v_x gives an invariant point; also there is one type of zero of v_x which varies with the energy constant h. Then when two zeros of v_x combine to form a multiple zero, two invariant points unite to form what it is natural to call a multiple invariant point. In general we may agree that an invariant point is multiple in the same sense in which the zeros of v_x are multiple. It can be shown that an invariant point which is multiple in the present sense is also multiple according to the definition of Birkhoff.

Let $x = x^*$ be a parallel trajectory and c^* the corresponding value of c. We can expand $F - c^{*2}$ in a Taylor's series about $(x^*, 0)$ and obtain

(35) $$v \cos^2 \phi - c^{*2} = \tfrac{1}{2}\left[v_{xx}(x^*)(x-x^*)^2 - 2v(x^*)\phi^2\right] + \cdots.$$

Now suppose that v has a minimum at $x = x^*$ with $v_{xx}(x^*) > 0$. Then

(35) shows that the path curve $F - c^{*2} = 0$ has two branches through $(x^*, 0)$ with distinct tangents there. From lemma 9 we see that T moves points toward the invariant point on one of the branches, and away from the invariant point on the other. Thus *an invariant point with $v_{xx}(x^*) > 0$ is unstable and of hyperbolic type*, and from lemma 9 we see at once that it is *directly unstable* [R 2, pp. 286-287; also R 3].

Now suppose that v has a maximum at $x = x^*$ with $v_{xx}(x^*) < 0$. Then (35) shows that the corresponding invariant point is surrounded by path curves of type C_3. Then *the invariant point is stable and of elliptic type.*

Finally, *the invariant points at which $v_{xx} = 0$ are multiple.* If v has a minimum at $x = x^*$, there are two branches of a path curve C_1 through the invariant point $(x^*, 0)$, but the two branches now have a common tangent. The invariant point is unstable. If v has a maximum at $x = x^*$, the invariant point is surrounded by path curves C_3 and is stable. If v has a point of inflection with a horizontal tangent, the path curve through the corresponding invariant point has a cusp there; the invariant point is unstable.

These results on stability may be compared with those yielded by equation (20).

F. Asymptotic, periodic, and recurrent trajectories. We shall now make use of the properties of T to determine the nature of the trajectories of the system. In the first place, the curves C_1 are made up of arcs terminated by invariant points. On each arc T is a translation toward one of the invariant points in one sense of the time and toward the other (which may be identical with the first) in the opposite sense. The invariant points represent parallel trajectories. The other points of the arc represent trajectories which are asymptotic to these parallels in the two senses of the time.

From lemma 5 we see that the path curves C_2 correspond to trajectories on which neither x' nor y' ever changes sign, i. e., the trajectories wind constantly about S in one direction. On the other hand, the path curves C_3 correspond to trajectories which oscillate between two parallels on S as we see from lemma 7. The two parallels correspond to the maximum and minimum values of x on the path curve. In this connection it is to be remembered that each path curve is merely a section of a cylindrical manifold in the manifold of states of motion on which the stream lines lie.

We shall now consider the periodic trajectories other than the parallel trajectories. On the path curves C_2 and C_3 the transformation T is a rotation $\tau' = \tau + \alpha$. Now if α has the form $2p\pi/q$, where p, q are integers without common factors, then T^q rotates the curve through p complete revolutions,

and every point is transformed into itself. The points on such curves then represent closed, periodic trajectories. These trajectories will be said to be of type $k:(p,q)$ where $k=2$ or 3 is the subscript of the corresponding path curve. The numbers have the following geometric significance. The number $q>0$ is the number of multiples of 2π by which y increases along the trajectory before it closes. For a trajectory of type $2:(p,q)$, $p>0$ $(p<0)$ is the number of multiples of ω by which x increases (decreases) along the trajectory; for a trajectory of type $3:(p,q)$, $p>0$ is the number of complete oscillations between two parallels. Furthermore, it is a natural extension to consider the meridians as trajectories of type $2:(p,0)$ for p any positive or negative integer not zero.

THEOREM 1. *Through each point of S there passes one and only one closed periodic trajectory of type $2:(p,q)$ where $p\neq 0$, $q\geqq 0$.*

If the theorem is true for the points of S on $y=0$, it is true for all the points of S. The trajectories of type $2:(p,q)$ through points on $y=0$ are represented on R by the points of the path curves C_2. Now by lemma 6 there is one and only one curve C_2 on which α takes on any given value, not zero; the existence of one and only one trajectory of type $2:(p,q)$ with $q>0$ is thus established. But the meridian $y=0$ corresponds to the trajectories with $q=0$. The theorem is proved. In this connection compare Birkhoff's existence theorem for closed trajectories of minimum type [R 2, p. 219].

The trajectories which correspond to the points on a path curve C_1, C_2, or C_3 will be called a *family of trajectories.* From lemma 8 we obtain at once the following theorem.

THEOREM 2. *Represented among the path curves C_3 which form a ring region there are at least two families of closed periodic trajectories of each type $3:(p,q)$ where $0<2p\pi/q<\alpha_M$. Represented among the path curves C_3 which form a circular region there is at least one family of closed periodic trajectories of each type $3:(p,q)$ where $0<2p\pi/q<\alpha_P$, and if $\alpha_M>\alpha_P$, at least two of each type $3:(p,q)$ where $\alpha_P<2p\pi/q<\alpha_M$.*

There remain to be considered only the trajectories represented by path curves C_2 and C_3 on which α is an irrational multiple of 2π. We shall now show that these trajectories are of the type known as *recurrent.* Instead of going into the details of defining recurrent trajectories, we state the following theorem which gives their characteristic property [R 7, p. 312]: *The necessary and sufficient condition that a stable trajectory be recurrent is that given any $\epsilon>0$ it is possible to find a t^* such that in any time interval of length t^* the trajectory comes within a distance ϵ of every point of the entire*

trajectory. In this theorem, as well as in the following one, the trajectory is considered as a stream line in (x, y, ϕ) space. Every trajectory of our system is stable in the sense in which the term is used in the theorem.

THEOREM 3. *The path curves C_2 and C_3 on which α is an irrational multiple of 2π represent recurrent trajectories.*

Consider two trajectories with their initial states of motion represented by two points τ_1 and τ_2 on a curve C_2 or C_3, on C^*_2 say. Now since the trajectories vary analytically with the initial conditions, it is possible to find an η such that when $|\tau_2 - \tau_1| < \eta$, the distance between the points of the trajectories for corresponding values of t is less than ϵ, at least until the next succeeding intersection of the trajectories with the surface of section. Let τ_0 be any point of C^*_2 and $\tau_1, \tau_2, \cdots$ its transforms under $T, T^2, \cdots$. Because of the nature of α, it is possible to find an N such that $|\tau - \tau_j| < \eta$, where τ is any point of C^*_2 and τ_j is some point of $\tau_0, \tau_1, \cdots, \tau_N$. Now it is possible to find a t^* such that any trajectory represented by a point of C^*_2 crosses the surface of section N times in any time interval of length t^*. It follows that any trajectory represented by a point of C^*_2 comes within a distance ϵ of every point of all trajectories represented by points on C^*_2 in any interval of time of length t^*. Hence, the trajectory is recurrent, and the proof is complete.

As remarked by Birkhoff, the recurrent motions of an integrable dynamical system are of the type known as *continuous.* Morse has treated *discontinuous* recurrent trajectories [R 8 and 9].

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REFERENCES.

1. Birkhoff, "Dynamical Systems," *American Mathematical Society Colloquium Publications* No. 9.

2. Birkhoff, "Dynamical Systems with Two Degrees of Freedom," *Transactions of the American Mathematical Society,* Vol. 18 (1917), pp. 199-300.

3. Birkhoff, "Surface Transformations and Their Dynamical Applications," *Acta Mathematica,* Vol. 43 (1922), pp. 1-119.

4. Birkhoff, "Proof of Poincaré's Geometric Theorem," *Transactions of the American Mathematical Society,* Vol. 14 (1913), pp. 14-22.

5. Appell, *Traité de Mécanique Rationnelle,* Vol. 3 (1921).

6. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste,* Vols. 1, 3.

7. Birkhoff, "Quelques Théorèmes sur le Mouvement des Systèmes Dynamiques," *Bulletin de la Société Mathématique de France,* Vol. 40 (1912), pp. 305-323.

8. Morse, "A One-to-One Representation of Geodesics on a Surface of Negative Curvature," *American Journal of Mathematics,* Vol. 43 (1921), pp. 33-51.

9. Morse, "Recurrent Geodesics on a Surface of Negative Curvature," *Transactions of the American Mathematical Society,* Vol. 22 (1921), pp. 84-100.

A BOUNDARY VALUE PROBLEM ASSOCIATED WITH THE CALCULUS OF VARIATIONS.†

By William T. Reid.

1. Introduction. Let η denote a set of variables $(\eta_1, \eta_2, \cdots, \eta_n)$ each of which is a function of the real variable x and denote the end values of these functions at x_1 and x_2 by $\eta(x_1)$ and $\eta(x_2)$. Let $\omega(x, \eta, \eta')$ be for each x on the interval x_1x_2 a homogeneous quadratic form in the variables η_i, η'_i, and denote by $H[\eta(x_1), \eta(x_2)]$ and $G[\eta(x_1), \eta(x_2)]$ two homogeneous quadratic forms in the variables $\eta_i(x_1)$, $\eta_i(x_2)$. Now consider the problem of minimizing the expression

$$I[\eta] = 2H[\eta(x_1), \eta(x_2)] + \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')\,dx \tag{1.1}$$

in the class of arcs

$$\eta_i = \eta_i(x) \qquad x_1 \leqq x \leqq x_2 \qquad\qquad (i = 1, 2, \cdots, n) \tag{1.2}$$

which satisfy a set of ordinary linear differential equations of the first order

$$\Phi_\alpha(x, \eta, \eta') = 0 \qquad\qquad (\alpha = 1, 2, \cdots, m < n), \tag{1.3}$$

which satisfy end-conditions

$$\Psi_\gamma[\eta(x_1), \eta(x_2)] = 0 \qquad\qquad (\gamma = 1, 2, \cdots, p \leqq 2n), \tag{1.4}$$

where the functions Ψ_γ are homogeneous of the first degree in their arguments, and which give a fixed value to the expression

$$2G[\eta(x_1), \eta(x_2)] + \int_{x_1}^{x_2} \eta_i(x) K_{ij}(x) \eta_j(x)\,dx \tag{1.5}$$

This problem of the calculus of variations may be reduced to a problem of Mayer of the type considered by Bliss.‡ The boundary value problem consisting of the Euler-Lagrange equations and the transversality conditions for this problem of the calculus of variations will be treated in the present paper.

In § 2 will be stated the hypotheses upon which the analysis is based, and in §§ 3 and 4 some properties of the boundary value problem are dis-

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† The present paper is the revised form of a paper written while the author was a National Research Fellow, and which was presented to the American Mathematical Society, September 9, 1931.

‡ G. A. Bliss, *Transactions of the American Mathematical Society*, Vol. 19 (1918), pp. 305-314.

cussed. In § 5 there are defined the successive classes S_i $(i = 1, -1, 2, -2, \cdots)$ of arcs $\eta(x)$ in which we shall consider the problem of minimizing $I[\eta]$. It is proven that if the class S_i is not empty then the greatest lower bound of $I[\eta]$ in this class is the absolute value of a characteristic number for the boundary value problem. In § 6 there are stated sufficient conditions for the boundary value problem to have infinitely many characteristic numbers. Finally, § 7 is devoted to some expansion theorems in terms of the characteristic solutions of our boundary value problem for the special case when the coefficients of $G[\eta(x_1), \eta(x_2)]$ are zero.

The problem defined above includes as special cases many boundary value problems which have previously been considered. If $\| K_{ij}(x) \|$ is the unit matrix and the coefficients of $H[\eta(x_1), \eta(x_2)]$ and $\Phi_a(x, \eta, \eta')$ are zero, while the end-conditions (1.4) reduce to

$$\eta_i(x_1) = 0 = \eta_i(x_2) \qquad (i = 1, 2, \cdots, n),$$

the boundary value problem is of the type considered by Hickson.† The proof of the existence of successive characteristic numbers for our boundary value problem in § 5 is by the direct methods of the calculus of variations and closely parallels the method used by Mason ‡ in treating second order differential equations. The method, therefore, is quite different from that used by Hickson in treating the special problem stated above. It should be mentioned, however, that if the differential system satisfies certain normality conditions which are more stringent than those imposed here, then the proof of the existence of successive characteristic numbers may be made in a manner entirely analogous to that used by Hickson.

For the general problem of Bolza with variable end-points the second variation is expressible in the form (1.1), where the variables η are the variations of a one-parameter family of admissible arcs, together with the variations of the end-points of such a family.§ Morse ¶ has obtained sufficient conditions for the problem of Bolza when the non-tangency condition is satisfied, in which case the variations of the end-points of the family may be expressed in terms of the variations of the functions of the family. He has defined for this problem of Bolza the accessory boundary value problem

† A. O. Hickson, *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 563-579.

‡ M. Mason, *The New Haven Colloquium of the American Mathematical Society*, pp. 173-222; in particular, pp. 207-222.

§ T. F. Cope, *Dissertation*, University of Chicago, 1927.

¶ M. Morse, *American Journal of Mathematics*, Vol. 53 (1931), pp. 517-546. This paper will be referred to as M.

as the above defined problem with $\| K_{ij} \|$ the identity matrix and the coefficients of $G[\eta(x_1), \eta(x_2)]$ all zero. The condition that the smallest characteristic number of this boundary value problem be positive plays in the sufficiency theorem for the problem of Bolza considered by Morse the same rôle that the requirement that the Jacobi condition be satisfied in the strong sense plays in sufficiency theorems for the simpler problems of the calculus of variations.† Morse has shown that there exist infinitely many characteristic numbers for this accessory boundary value problem. The proof of this existence theorem is quite different from the proof of the existence theorem for the general boundary value problem here discussed, and depends upon the results obtained by Morse on the calculus of variations in the large.‡

2. Notation and preliminary remarks. In the later discussion the following subscripts will have the ranges indicated: $i, j, k = 1, 2, \cdots, n$; $\alpha = 1, 2, \cdots, m$; $\pi, \kappa, \tau = 1, 2, \cdots, 2n$; $\gamma = 1, 2, \cdots, p$; $\theta = 1, 2, \cdots, 2n - p$. The repetition of a subscript in a single term of an expression will denote summation with respect to that subscript over the range on which it is defined. Partial derivatives of ω and Φ_α with respect to the variables η_i and η'_i will be denoted by writing the variable as a subscript; correspondingly, the derivatives of the functions Ψ_γ, H and G with respect to $\eta_i(x_1)$ and $\eta_i(x_2)$ will be denoted by the subscripts η_{i1} and η_{i2} respectively.

An arc $\eta \equiv [\eta_i(x)]$ will be called *differentially admissible* if the functions $\eta_i(x)$ are of class D' § on x_1x_2 and satisfy (1.3) on this interval. An arc which satisfies the end-conditions (1.4) will be said to be *terminally admissible*. Finally, an arc which is both differentially and terminally admissible will be called *admissible*.¶

The analysis is based upon the following hypotheses:

(H1) All the coefficients in (1.1), (1.3), (1.4) and (1.5) are real, and the functions $\omega_{\eta'_i\eta'_j}$, $\omega_{\eta'_i\eta_j}$, $\Phi_{\alpha\eta'_j}$ are of class C', while the functions

† Recently Bliss has given sufficient conditions for the problem of Bolza for the more general case in which the non-tangency condition is not assumed to be satisfied. See G. A. Bliss, *Annals of Mathematics*, Vol. 33 (1932), pp. 261-274. Instead of stating the analogue of the Jacobi condition in terms of the smallest characteristic number of a boundary value problem involving differential equations and end-conditions, Bliss has replaced this boundary value problem by a corresponding algebraic problem.

‡ M. Morse, *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 379-404.

§ We shall use the terminology for classes of functions introduced by Bolza. See O. Bolza, *Lectures on the Calculus of Variations*, 1904, p. 7.

¶ This terminology is due to Morse. See M, p. 518.

$\omega_{\eta_i\eta_j}$, $\Phi_{a\eta_j}$ and K_{ij} are continuous on x_1x_2. The matrices $\| \omega_{\eta'_i\eta'_j} \|$, $\| \omega_{\eta_i\eta_j} \|$ and $\| K_{ij} \|$ are symmetric and $\| \Phi_{a\eta'_j} \|$ is of rank m at each point on x_1x_2, while the constant matrix $\| \Psi_{\gamma\eta_{i1}}; \Psi_{\gamma\eta_{i2}} \|$ has rank p.

(H2) The symmetric matrix

$$\left\| \begin{matrix} \omega_{\eta'_i\eta'_j} & \Phi_{a\eta'_i} \\ \Phi_{a\eta'_j} & 0 \end{matrix} \right\| \tag{2.1}$$

is non-singular on x_1x_2.

If η is a normal minimizing arc for the problem of the calculus of variations defined in § 1, it follows that there exists a constant λ and functions $\mu_a(x)$ such that if we define

$$\Omega(x, \eta, \eta', \mu) = \omega(x, \eta, \eta') + \mu_a(x)\Phi_a(x, \eta, \eta') \tag{2.2}$$

and

$$J_i(\eta, \mu) = d\Omega_{\eta'_i}/dx - \Omega_{\eta_i}, \tag{2.3}$$

then on every sub-arc between corners of η the differential equations

$$J_i(\eta, \mu) + \lambda K_{ij}\eta_j = 0, \qquad \Phi_a(x, \eta, \eta') \equiv \Phi_{a\eta'_i}(x)\eta'_i + \Phi_{a\eta_i}(x)\eta_i = 0 \tag{2.4}$$

are satisfied; furthermore, there exist constants d_γ such that

$$\begin{aligned} &L_{i1}(\eta, \mu, d, \lambda) \\ &\qquad \equiv H_{\eta_{i1}}[\eta] + d_\gamma\Psi_{\gamma\eta_{i1}} - \lambda G_{\eta_{i1}}[\eta] - \Omega_{\eta'_i}(x, \eta, \eta', \mu) \,\Big|^{x=x_1} = 0, \\ &L_{i2}(\eta, \mu, d, \lambda) \\ &\qquad \equiv H_{\eta_{i2}}[\eta] + d_\gamma\Psi_{\gamma\eta_{i2}} - \lambda G_{\eta_{i2}}[\eta] + \Omega_{\eta'_i}(x, \eta, \eta', \mu) \,\Big|^{x=x_2} = 0, \\ &\Psi_\gamma[\eta(x_1), \eta(x_2)] \equiv \Psi_{\gamma\eta_{i1}}\eta_i(x_1) + \Psi_{\gamma\eta_{i2}}\eta_i(x_2) = 0. \end{aligned} \tag{2.5}$$

The boundary value problem (2.4), (2.5) is the one considered in this paper. A constant λ will be said to be a characteristic number if for this value there exist functions $\eta_i(x)$ of class C'' with multipliers $\mu_a(x)$ of class C' such that the set η_i, μ_a does not vanish identically on x_1x_2, the set satisfies (2.4) on this interval, and is such that there exist constants d_γ with which the end-values of the set satisfy (2.5).

(H3) The only solution η_i, μ_a of the system (2.4), (2.5) for which $\eta_i \equiv 0$ on x_1x_2 is the identically vanishing solution $\eta_i \equiv 0 \equiv \mu_a$ on x_1x_2. This hypothesis is a condition of normality on the interval x_1x_2 with respect to the differential equations (1.3) and the conditions (2.5), independent of the quantities $\| K_{ij} \|$ and $G[\eta(x_1), \eta(x_2)]$.†

† For a discussion of normality conditions for the problem of Lagrange, see G. A. Bliss, *American Journal of Mathematics*, Vol. 52 (1930), pp. 673-744; in particular, pp. 687-695. This paper will be referred to as B. I.

(H4) $I[\eta] > 0$ for every admissible arc η which is not identically zero on x_1x_2.

With respect to the boundary value problem (2.4), (2.5), condition (H4) is not as restrictive as it may first seem to be. If there is a constant λ_0 such that the expression

$$I[\eta] + \lambda_0\{2G[\eta(x_1), \eta(x_2)] + \int_{x_1}^{x_2} \eta_i K_{ij}\eta_j \, dx\}$$

is positive for all non-identically vanishing admissible arcs, then $I[\eta]$ may be replaced by this expression. The modified boundary value problem is equivalent to (2.4), (2.5), and may be reduced to it by a linear change of parameter.

3. Properties of the boundary value problem (2.4), (2.5). In this section will be given some fundamental properties of the differential system (2.4), (2.5). For brevity $H[\eta]$ will be written for $H[\eta(x_1), \eta(x_2)]$ and $H[u;v]$ will be used to denote $H_{\eta_{i1}}[u]v_i(x_1) + H_{\eta_{i2}}[u]v_i(x_2)$; corresponding notations will be introduced for $G[\eta(x_1), \eta(x_2)]$. We have the fundamental identities

$$(3.1) \qquad \begin{aligned} H[u;\, v] &\equiv H[v;\, u], \qquad & H[u;\, u] &\equiv 2H[u], \\ G[u;\, v] &\equiv G[v;\, u], \qquad & G[u;\, u] &\equiv 2G[u]. \end{aligned}$$

Similarly, for the quadratic form $\Omega(x, \eta, \eta', \mu)$ we have:

$$(3.2) \qquad 2\Omega(x, \eta, \eta', \mu) \equiv \eta'_i\Omega_{\eta'_i} + \eta_i\Omega_{\eta_i} + \mu_a\Omega_{\mu_a},$$

$$(3.3) \qquad u'_i\Omega_{v'_i} + u_i\Omega_{v_i} + \rho_a\Omega_{\sigma_a} \equiv v'_i\Omega_{u'_i} + v_i\Omega_{u_i} + \sigma_a\Omega_{\rho_a},$$

where the derivatives of Ω have the arguments (η, η', μ), (u, u', ρ) or (v, v', σ) as indicated by their subscripts.

It is also to be noted that if u_i, ρ_a is an arbitrary set of functions and d_γ and λ are arbitrary constants, then for every terminally admissible arc v we have

$$(3.4) \qquad \begin{aligned} &v_i(x_1)L_{i1}(u, \rho, d, \lambda) + v_i(x_2)L_{i2}(u, \rho, d, \lambda) \\ &\qquad \equiv H[u;\, v] - \lambda G[u;\, v] + v_i(x)\Omega_{u'_i}(x, u, u', \rho) \Big|_1^2 . \end{aligned}$$

In particular, if u_i, ρ_a is a set which satisfies (2.5) with constants d_γ and λ, then for every terminally admissible arc v we have

$$(3.4') \qquad 0 = H[u;\, v] - \lambda G[u;\, v] + v_i(x)\Omega_{u'_i}(x, u, u', \rho) \Big|_1^2 .$$

If u and v are both differentially admissible arcs of class C'' and ρ_a is an arbitrary set of functions of class C', it follows from an integration by parts that

$$(3.5)\quad \int_{x_1}^{x_2} v_i J_i(u,\rho)\,dx = v_i\Omega_{u'_i}(x,u,u',\rho)\,\Big|_1^2 - \int_{x_1}^{x_2}[v'_i\Omega_{u'_i} + v_i\Omega_{u_i}]\,dx.$$

It follows by the use of (3.2), (3.4′) and (3.5) that

LEMMA 3.1. *If η_i, μ_a is a solution of the boundary value problem* (2.4), (2.5) *corresponding to a value λ, then the expression*

$$(3.6)\qquad I[\eta\mid\lambda] \equiv I[\eta] - \lambda\{2G[\eta] + \int_{x_1}^{x_2}\eta_i K_{ij}\eta_j\,dx\}$$

has the value zero.

COROLLARY. *The value $\lambda = 0$ is not a characteristic number of* (2.4), (2.5).

LEMMA 3.2. *If u_i, ρ_a and v_i, σ_a are solutions of* (2.4), (2.5) *corresponding to distinct characteristic numbers λ and λ^*, then*

$$G[u;\,v] + \int_{x_1}^{x_2} u_i K_{ij} v_j\,dx = 0.$$

This lemma follows directly from the relation

$$(\lambda - \lambda^*)\int_{x_1}^{x_2} u_i K_{ij} v_j dx = \int_{x_1}^{x_2}[u_i J_i(v,\sigma) - v_i J_i(u,\rho)]dx$$

and the relations (3.3), (3.4′) and (3.5).

LEMMA 3.3. *The boundary value problem* (2.4), (2.5) *has only real characteristic numbers.*

For suppose that η_i, μ_a is a solution of (2.4), (2.5) corresponding to a complex value λ. Then $\bar\eta_i$, $\bar\mu_a$, the conjugate imaginaries of the set η_i, μ_a, furnish a solution of this system corresponding to $\bar\lambda$, the conjugate imaginary of λ. By Lemma 3.2,

$$(3.7)\qquad G[\bar\eta;\,\eta] + \int_{x_1}^{x_2}\bar\eta_i K_{ij}\eta_j\,dx = 0.$$

Furthermore,

$$0 = \int_{x_1}^{x_2}\bar\eta_i[J_i(\eta,\mu) + \lambda K_{ij}\eta_j]\,dx$$

$$(3.8)\qquad = -H[\bar\eta;\,\eta] - \int_{x_1}^{x_2}[\bar\eta'_i\Omega_{\eta'_i} + \bar\eta_i\Omega_{\eta_i}]\,dx,$$

in view of (3.4′), (3.5) and (3.7). If u and v are the real and pure imaginary parts of η, it is readily shown that the right member of (3.8) reduces to $-I[u] - I[v]$, and therefore by (H4) we have that $u \equiv 0 \equiv v$, and hence $\eta \equiv 0$ on x_1x_2. But from (H3) it would then follow that the multipliers μ_a are all zero on x_1x_2, and therefore this value of λ is not a

characteristic number. Hence (2.4), (2.5) has only real characteristic numbers and the corresponding characteristic solutions may be chosen real.

4. The canonical form of the differential system (2.4), (2.5). In the equations (2.4) one may introduce the canonical variables

$$\zeta_i = \Omega_{\eta'_i}(x, \eta, \eta', \mu) \qquad (i = 1, 2, \cdots, n). \tag{4.1}$$

Since the matrix (2.1) is non-singular on x_1x_2, the system (1.3), (4.1) of $m + n$ equations has solutions

$$\eta'_i = \chi_i(x, \eta, \zeta), \qquad \mu_a = M_a(x, \eta, \zeta)$$

which are linear in η_i, ζ_i. Then the equations (2.4) are equivalent to

$$\eta'_i = \chi_i(x, \eta, \zeta), \quad \zeta'_i = \Omega_{\eta_i}[x, \eta, \chi(x, \eta, \zeta), M(x, \eta, \zeta)] - \lambda K_{ij}\eta_j. \tag{2.4'}$$

If $a_{\theta i}, b_{\theta i}$ $(\theta = 1, 2, \cdots, 2n - p)$ are linearly independent solutions of

$$\Psi_{\gamma\eta_{i1}}a_{\theta i} + \Psi_{\gamma\eta_{i2}}b_{\theta i} = 0 \qquad (\gamma = 1, 2, \cdots, p),$$

then (2.5) is equivalent to the $2n$ linearly independent relations

$$\begin{aligned} a_{\theta i}\{H_{\eta_{i1}} - \lambda G_{\eta_{i1}} - \zeta_i(x_1)\} + b_{\theta i}\{H_{\eta_{i2}} - \lambda G_{\eta_{i2}} + \zeta_i(x_2)\} &= 0 \\ \Psi_\gamma[\eta(x_1), \eta(x_2)] &= 0 \end{aligned} \tag{2.5'}$$

in the end-values of the variables η_i, ζ_i. The boundary value problem (2.4), (2.5) is then equivalent to the boundary value problem (2.4′), (2.5′) in the $2n$ variables η_i, ζ_i.

There will now be given some properties of the differential system (2.4′), (2.5′) which will be used in the later sections. One might proceed at once to these sections and then, whenever the results of this section are used, refer back to the present discussion.

To obtain compactness of notation we note that system (2.4′), (2.5′) is of the form

$$\begin{gathered} y'_\pi = (A_{\pi\tau} + \lambda B_{\pi\tau})y_\tau, \quad s_\pi(\lambda: y) \equiv M_{\pi\tau}(\lambda)y_\tau(x_1) + N_{\pi\tau}(\lambda)y_\tau(x_2) = 0, \\ (\pi, \tau = 1, 2, \cdots, 2n), \end{gathered} \tag{4.2}$$

where $y_i = \eta_i$, $y_{n+i} = \zeta_i$ $(i = 1, 2, \cdots, n)$. The functions $A_{\pi\tau}$ and $B_{\pi\tau}$ are continuous on x_1x_2, while $M_{\pi\tau}$ and $N_{\pi\tau}$ are linear in λ and the matrix $\| M_{\pi\tau}(\lambda), N_{\pi\tau}(\lambda) \|$ has rank $2n$ for each value of λ. The general system (4.2) is of the type discussed by Bliss.† Let $Y(x, \lambda) \equiv \| Y_{\pi\tau}(x, \lambda) \|$ be

† G. A. Bliss, *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 561-584. This paper will be referred to as B. II. In the system treated by Bliss $M_{\pi\tau}$ and $N_{\pi\tau}$ are supposed to be independent of λ, but the properties of the above system (4.2) that we state here may be proven by the same methods that Bliss uses.

a matrix whose columns are $2n$ linearly independent solutions of (4.2) and which reduces to the unit matrix for $x = x_1$. System (4.2) has exactly r linearly independent solutions for a given value λ if and only if the matrix $\| s_\pi[\lambda : Y_\tau(x, \lambda)] \|$ has rank $2n - r$. Now $| s_\pi[\lambda : Y_\tau(x, \lambda)] |$ is a permanently convergent power series in λ and its zeros are the characteristic numbers of (4.2). For the particular system (2.4′), (2.5′) we have that $\lambda = 0$ is not a characteristic number and therefore the corresponding determinant $| s_\pi[\lambda : Y_\tau(x, \lambda)] |$ is not identically zero. Hence for this system we have

LEMMA 4.1. *The totality of characteristic numbers of (2.4′), (2.5′) is denumerable and has no finite accumulation point.*

The system adjoint to (4.2) is

$$(4.3) \quad z'_\pi = -(A_{\tau\pi} + \lambda B_{\tau\pi}) z_\tau, \quad t_\pi(\lambda : z) \equiv P_{\tau\pi}(\lambda) z_\tau(x_1) + Q_{\tau\pi}(\lambda) z_\tau(x_2) = 0,$$

where $P_{\tau\pi}$ and $Q_{\tau\pi}$ are such that the matrix of coefficients in the boundary conditions of (4.3) is of rank $2n$ and $M_{\pi\kappa}(\lambda) P_{\kappa\tau}(\lambda) - N_{\pi\kappa}(\lambda) Q_{\kappa\tau}(\lambda) = 0$ $(\pi, \tau = 1, 2, \cdots, 2n)$. For a given value of λ the number of linearly independent solutions is the same for (4.2) and (4.3).

If we define

$$s^*_\pi(\lambda : y) = M^*_{\pi\tau}(\lambda) y_\tau(x_1) + N^*_{\pi\tau}(\lambda) y_\tau(x_2)$$
$$t^*_\pi(\lambda : z) = P^*_{\tau\pi}(\lambda) z_\tau(x_1) + Q^*_{\tau\pi}(\lambda) z_\tau(x_2),$$

where $M^*_{\pi\tau}$, $N^*_{\pi\tau}$, $P^*_{\pi\tau}$, and $Q^*_{\pi\tau}$ are such that for each value of λ the matrices

$$\begin{Vmatrix} M_{\pi\tau} & N_{\pi\tau} \\ M^*_{\pi\tau} & N^*_{\pi\tau} \end{Vmatrix} \qquad \begin{Vmatrix} -P^*_{\pi\tau} & -P_{\pi\tau} \\ Q^*_{\pi\tau} & Q_{\pi\tau} \end{Vmatrix}$$

are reciprocals, then, as shown by Bliss, we have the identity

$$s_\pi(\lambda : y) t^*_\pi(\lambda : z) + s^*_\pi(\lambda : y) t_\pi(\lambda : z) = y_\pi(x) z_\pi(x) \Big|_{x=x_1}^{x=x_2}$$

If $| s_\pi[\lambda : Y_\tau(x, \lambda)] | \neq 0$ for a given λ, then for this value of λ the non-homogeneous system

$$(4.4) \quad y'_\pi = (A_{\pi\tau} + \lambda B_{\pi\tau}) y_\tau + g_\pi, \qquad s_\pi(\lambda : y) = h_\pi \quad (\pi = 1, 2, \cdots, 2n),$$

where the functions $g_\pi(x)$ are continuous and the h_π are constants, has a unique solution of class C'. If the functions $g_\pi(x)$ are merely of class D, then (4.4) has a unique solution which is continuous and whose derivatives are continuous except possibly at the discontinuities of the functions $g_\pi(x)$. If $\| s_\pi[\lambda : Y_\tau(x, \lambda)] \|$ has rank $2n - r$ for a given λ, then corresponding to this value of λ the system (4.4) has a solution if and only if the relation

$$\int_{x_1}^{x_2} z_\pi(x) g_\pi(x)\,dx = h_\pi t^*_\pi(\lambda : z) \tag{4.5}$$

is satisfied by every solution z of the adjoint system (4.3) for this value of λ.† The most general solution of (4.4) is then $y_\pi = y^*_\pi + c_1 Y_{\pi 1} + \cdots + c_r Y_{\pi r}$, where (y^*_π) is a particular solution and $(Y_{\pi 1}), \cdots, (Y_{\pi r})$ are r linearly independent solutions of (4.2).

Bliss has called a system (4.2) self-adjoint if the differential equations and the boundary conditions of its adjoint are equivalent to its own for all values of λ by means of a transformation $z_\pi = T_{\pi\tau}(x) y_\tau$, where the matrix $\| T_{\pi\tau} \|$ is non-singular and its elements are of class C' on x_1x_2. The system (2.4′), (2.5′) may be shown to be self-adjoint, and the matrix of transformation $\| T_{\pi\tau} \|$ is the constant matrix ‡

$$\begin{Vmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{Vmatrix}. \tag{4.6}$$

Suppose that corresponding to a given λ there are exactly r linearly independent solutions $(Y_{\pi 1}), \cdots, (Y_{\pi r})$ of (4.2). Then solutions $(Y_{\pi, r+1}), \cdots, (Y_{\pi, 2n})$ may be chosen such that $Y(x, \lambda) \equiv \| Y_{\pi\tau}(x, \lambda) \|$ is non-singular on x_1x_2. Let $Z(x, \lambda) \equiv \| Z_{\pi\tau}(x, \lambda) \|$ be the reciprocal of $Y(x, \lambda)$ on x_1x_2; then each row of $Z(x, \lambda)$ is a solution of (4.3). Now integers $i_1, i_2, \cdots, i_{2n-r}$ exist such that the matrix $\| s_{i\beta}[\lambda : Y_{r+\xi}(x, \lambda)] \|$ $(\beta, \xi = 1, 2, \cdots, 2n - r)$ has a unique reciprocal, which we will denote by $\| s_{\beta\xi}^{-1}(\lambda) \|$ $(\beta, \xi = 1, 2, \cdots, 2n - r)$. Now define the matrix $D(\lambda) \equiv \| D_{\pi\tau}(\lambda) \|$ as follows: $D_{r+\beta, i_\xi}(\lambda) = s_{\beta\xi}^{-1}(\lambda)$, $D_{\pi\tau}(\lambda) = 0$ if $\pi \leqq r$ or $\tau \neq i_\xi$ $(\beta, \xi = 1, 2, \cdots, 2n - r)$. Let

$$G_{\pi\tau}(x, t, \lambda) = \tfrac{1}{2} Y_{\pi\kappa}(x, \lambda) \left[\frac{|x - t|}{x - t} \delta_{\kappa\nu} + D_{\kappa v}(\lambda) \Delta_{v\nu}(\lambda) \right] Z_{\nu\tau}(t, \lambda), \tag{4.7}$$

where $Y_{\pi\tau}(x, \lambda)$, $Z_{\pi\tau}(x, \lambda)$ and $D_{\pi\tau}(\lambda)$ are defined as above and $\Delta_{\pi\tau}(\lambda) = M_{\pi\kappa}(\lambda) Y_{\kappa\tau}(x_1, \lambda) - N_{\pi\kappa}(\lambda) Y_{\kappa\tau}(x_2, \lambda)$. The subscripts ν and v which occur in (4.7) are supposed to have also the range $1, 2, \cdots, 2n$. If $g_\pi(x)$ and h_π are such that the system (4.4) has a solution for a given value of λ, then a particular solution of that system is given by

$$y_\pi(x, \lambda) = \int_{x_1}^{x_2} G_{\pi\tau}(x, t, \lambda) g_\tau(t)\,dt + Y_{\pi\kappa}(x, \lambda) D_{\kappa\tau}(\lambda)\, h_\tau.\S \tag{4.8}$$

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† For the case $h_\pi = 0$ $(\pi = 1, 2, \cdots, 2n)$ this result is proven by Bliss. See B. II, p. 567. The same method of proof applies to the more general case to give the stated result.

‡ See Cope, *loc. cit.*

§ If $r = 0$, then $\| G_{\pi\tau}(x, t, \lambda) \|$ is the ordinary Green's matrix and the result is

The matrix $\| G_{\pi\tau}(x, t, \lambda) \|$ for (2.4′), (2.5′) we will denote by

$$(4.9) \qquad \left\| \begin{matrix} G^1{}_{ij}(x,t,\lambda) & G^2{}_{ij}(x,t,\lambda) \\ G^3{}_{ij}(x,t,\lambda) & G^4{}_{ij}(x,t,\lambda) \end{matrix} \right\|,$$

where $\| G^1{}_{ij} \|$, $\| G^2{}_{ij} \|$, $\| G^3{}_{ij} \|$ and $\| G^4{}_{ij} \|$ are n-rowed square matrices. If we apply the above result to the non-homogeneous system

$$(4.10) \qquad \begin{aligned} &\eta'_i = \chi_i(x,\eta,\zeta), \quad \zeta'_i = \Omega_{\eta_i}[x,\eta,\chi(x,\eta,\zeta), M(x,\eta,\zeta)] - \lambda K_{ij}\eta_j + k_i, \\ &a_{\theta i}\{H_{\eta_{i1}} - \lambda G_{\eta_{i1}} - \zeta_i(x_1)\} + b_{\theta i}\{H_{\eta_{i2}} - \lambda G_{\eta_{i2}} + \zeta_i(x_2)\} = h_\theta, \\ &\Psi_\gamma[\eta(x_1), \eta(x_2)] = 0 \end{aligned}$$

corresponding to the homogeneous system (2.4′), (2.5′), we have that if this system has a solution for a given value of λ, then a particular solution is given by

$$(4.11) \qquad \begin{aligned} \eta_i(x,\lambda) &= \int_{x_1}^{x_2} G^2{}_{ij}(x,t,\lambda)k_j(t)\,dt + Y_{i\kappa}(x,\lambda)D_{\kappa\theta}(\lambda)h_\theta, \\ \zeta_i(x,\lambda) &= \int_{x_1}^{x_2} G^4{}_{ij}(x,t,\lambda)k_j(t)\,dt + Y_{n+i,\kappa}(x,\lambda)D_{\kappa\theta}(\lambda)h_\theta. \end{aligned}$$

For the differential system (2.4′), (2.5′) we have in view of the corollary to Lemma 3.1 that $\| G_{\pi\tau}(x, t, 0) \|$ is an ordinary Green's matrix. Since this system is self-adjoint and the matrix $\| T_{\pi\tau} \|$ is given by (4.6), we have †

$$(4.12) \qquad \begin{aligned} G^2{}_{ij}(x,t,0) = G^2{}_{ji}(t,x,0); \quad G^3{}_{ij}(x,t,0) &= G^3{}_{ji}(t,x,0); \\ G^1{}_{ij}(x,t,0) = -\,G^4{}_{ji}(t,x,0). \end{aligned}$$

From the form (4.11) for a solution of (4.10) if that system is compatible, we obtain at once that the differential system (2.4′), (2.5′) is equivalent to the following system of integral equations

$$(4.13) \qquad \begin{aligned} \eta_i(x) &= -\lambda \int_{x_1}^{x_2} G^2{}_{ik}(x,t,0)K_{kj}(t)\eta_j(t)\,dt + \lambda Y_{i\kappa}(x,0)D_{\kappa\theta}(0)h_\theta{}^0, \\ \zeta_i(x) &= -\lambda \int_{x_1}^{x_2} G^4{}_{ik}(x,t,0)K_{kj}(t)\eta_j(t)\,dt + \lambda Y_{n+i,\kappa}(x,0)D_{\kappa\theta}(0)h_\theta{}^0, \end{aligned}$$

where $h_\theta{}^0 = a_{\theta i}G_{\eta_{i1}} + b_{\theta i}G_{\eta_{i2}}$ $(\theta = 1, 2, \cdots, 2n - p)$.

Finally, we state a lemma which will be of use in § 7,

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proven by Bliss. See B. II. In this case the solution is unique. If $r > 0$, then $\| G_{\pi\tau}(x, t, \lambda) \|$ is a generalized Green's matrix and this result may be proven in the same manner as given in a recent paper by the author for the special case $h_\pi = 0$. See W. T. Reid, *American Journal of Mathematics*, Vol. 53 (1931), pp. 443-459; in particular, pp. 447, 448.

† See Bliss, B. II, p. 580,. The relations (4.12) hold true for any value of λ which is not a characteristic number of (2.4′), (2.5′), but we use the relation only for the special case $\lambda = 0$.

LEMMA 4.2. *If the functions $k_j(x)$ are of class D on x_1x_2, and*

$$\eta_i(x) = \int_{x_1}^{x_2} G^2{}_{ij}(x, t, 0) k_j(t) dt, \tag{4.14}$$

then $\eta \equiv [\eta_i(x)]$ is an admissible arc.

This lemma follows immediately since η_i, ζ_i, where η_i is given by (4.14) and $\zeta_i(x) = \int_{x_1}^{x_2} G^4{}_{ij}(x, t, 0) k_j(t) dt$, is a solution of (4.10) for $\lambda = 0$, $h_\theta = 0$ $(\theta = 1, 2, \cdots, 2n - p)$.

5. Existence of characteristic numbers for (2.4), (2.5). There will now be defined a sequence of classes of admissible arcs, in which we shall consider the problem of minimizing the expression $I[\eta]$. The class S_1 is defined as consisting of the totality of admissible arcs η which satisfy the relation

$$2G[\eta] + \int_{x_1}^{x_2} \eta_i K_{ij} \eta_j dx = 1. \tag{5.1}$$

The sequence of classes is defined by induction as follows: Suppose classes $S_1, S_2, \cdots, S_{s-1}$ $(s \geqq 2)$ have been defined and are not empty, and for $\lambda = \lambda_t$ $(t = 1, 2, \cdots, s - 1)$, where λ_t is the greatest lower bound of $I[\eta]$ in the class S_t, there are $r_t (0 < r_t \leqq 2n)$ linearly independent solutions of (2.4), (2.5). Let $\eta_i{}^\beta$, $\mu_a{}^\beta$ $(\beta = 1, 2, \cdots, r_1 + r_2 + \cdots + r_{s-1})$ denote these characteristic solutions of (2.4), (2.5), and let λ_β denote the value of λ corresponding to the solution $\eta_i{}^\beta$, $\mu_a{}^\beta$. The class S_s is then defined as the totality of arcs η of class S_{s-1} which satisfy the relations

$$G[\eta^\beta; \eta] + \int_{x_1}^{x_2} \eta_i{}^\beta K_{ij} \eta_j dx = 0 \qquad (\beta = 1, 2, \cdots, r_1 + \cdots + r_{s-1}). \tag{5.2}$$

There will be proven in this section the following theorem:

THEOREM 5.1. *If the class S_s is not empty and λ_s is the greatest lower bound of $I[\eta]$ in this class, then $\lambda = \lambda_s$ is a characteristic number of* (2.4), (2.5) *and $\lambda_s > \lambda_{s-1}$.*

In particular, the proof of this theorem applies to the case $s = 1$, when λ_0 is set equal to zero. By the corollary to Lemma 3.1 we know that $\lambda = 0$ is not a characteristic number of (2.4), (2.5), and we therefore obtain from Theorem 5.1 a complete induction proof for the existence of successive characteristic numbers for our boundary value problem.

Theorem 5.1 will be established by showing that if we assume it is false we are led to a contradiction. Clearly $\lambda_s \geqq \lambda_{s-1}$. If the theorem is false,

then either $\lambda_s > \lambda_{s-1}$ and is not a characteristic number of (2.4), (2.5), or $\lambda_s = \lambda_{s-1}$. Now let

$$(5.3) \qquad u_\nu \equiv (u_{i\nu}) \qquad (\nu = 1, 2, \cdots)$$

be a sequence of arcs of class S_s such that $\lim_{\nu\to\infty} I[u_\nu] = \lambda_s$. On the assumption that S_s is not empty such a sequence clearly exists. The following auxiliary lemma will now be proven.

LEMMA 5.1. *If Theorem 5.1 is false, then the non-homogeneous system*

$$(5.4) \qquad J_i(\eta, \mu) + \lambda_s K_{ij}\eta_j = K_{ij}u_{j\nu}, \qquad \Phi_a(x, \eta, \eta') = 0,$$

$$(5.5) \qquad L_{i1}(\eta, \mu, d, \lambda_s) + G\eta_{i1}[u_\nu] = 0$$
$$= L_{i2}(\eta, \mu, d, \lambda_s) + G\eta_{i2}[u_\nu] = \Psi_\gamma[\eta(x_1), (x_2)]$$

where u_ν is defined as above, has a solution $v_{i\nu}$, $\sigma_{a\nu}$ such that the arc v_ν satisfies the relations (5.2), *and furthermore,*

$$(5.6) \qquad |\, v_{i\nu}(x) \,| \leqq l_1 \sum_{i=1}^{n} \{ \int_{x_1}^{x_2} u^2{}_{i\nu}dx + u^2{}_{i\nu}(x_1) + u^2{}_{i\nu}(x_2)\} + l_2 \qquad (\nu = 1, 2, \cdots)$$

on x_1x_2, where l_1 and l_2 are constants independent of x and ν.

By the introduction of the canonical variables (4.1) the system (5.4), (5.5) is reduced to the system (4.10), with $\lambda = \lambda_s$, $k_i = K_{ij}u_{j\nu}$ and $h_\theta = -\{G\eta_{i1}[u_\nu]a_{\theta i} + G\eta_{i2}[u_\nu]b_{\theta i}\}$. If $\lambda_s > \lambda_{s-1}$ and is not a characteristic number of (2.4), (2.5), then there exists a unique solution of (5.4), (5.5); if $\lambda_s = \lambda_{s-1}$, since the arcs u_ν satisfy (5.2) it may be shown that the functions $k_i(x)$, h_θ defined above satisfy for the system (4.10) the condition (4.5) for the value $\lambda = \lambda_s$, and hence the system (4.10) is still compatible. It then follows that for $\lambda = \lambda_s$, $k_i = K_{ij}u_{j\nu}$, $h_\theta = -\{G\eta_{i1}[u_\nu]a_{\theta i} + G\eta_{i2}[u_\nu]b_{\theta i}\}$ the system (4.10) has a particular solution $\eta^0{}_{i\nu}$, $\zeta^0{}_{i\nu}$ given by the relations (4.11), and by the application of elementary inequalities it is seen that the functions $\eta^0{}_{i\nu}(x)$ satisfy an inequality of the form (5.6). Now if $\sigma^0{}_{a\nu} = M_a(x, \eta_\nu{}^0, \zeta_\nu{}^0)$, we have

$$(\lambda_s - \lambda_\beta) \int_{x_1}^{x_2} \eta^0{}_{i\nu}K_{ij}\eta_j{}^\beta dx$$
$$= \int_{x_1}^{x_2} [\eta^0{}_{i\nu}J_i(\eta^\beta, \mu^\beta) - \eta_i{}^\beta J_i(\eta_\nu{}^0, \sigma_\nu{}^0) + \eta_i{}^\beta K_{ij}u_{j\nu}]dx,$$

and since the arc u_ν is of class S_s, it follows in view of (3.3), (3.4) and (3.5) that

$$(5.7) \qquad (\lambda_s - \lambda_\beta)\{ \int_{x_1}^{x_2} \eta^0{}_{i\nu}K_{ij}\eta_j{}^\beta dx + G[\eta_\nu{}^0;\, \eta^\beta]\} = 0 \qquad (\beta = 1, 2, \cdots, r_1 + \cdots + r_{s-1}).$$

Now set

$$(5.8)\qquad \begin{aligned} \eta^*_{i\nu} &= \eta^0_{i\nu} - \sum_{\beta=1}^{R} \{ \int_{x_1}^{x_2} \eta^0_{i\nu} K_{ij} \eta_j{}^\beta dx + G[\eta_\nu{}^0;\ \eta^\beta]\} \eta_i{}^\beta, \\ \zeta^*_{i\nu} &= \zeta^0_{i\nu} - \sum_{\beta=1}^{R} \{ \int_{x_1}^{x_2} \eta^0_{i\nu} K_{ij} \eta_j{}^\beta dx + G[\eta_\nu{}^0;\ \eta^\beta]\} \zeta_i{}^\beta, \end{aligned}$$

where $R = r_1 +, \cdots, r_{s-1}$. If $\lambda_s > \lambda_{s-1}$, in view of (5.7) we have that $\eta^*_{i\nu}, \zeta^*_{i\nu} \equiv \eta^0_{i\nu}, \zeta^0_{i\nu}$. If $\lambda_s = \lambda_{s-1}$, then by (5.7) the coefficients of $\eta_i{}^\beta(x)$ and $\zeta_i{}^\beta(x)$ in (5.8) are zero unless $\lambda_\beta = \lambda_{s-1}$. In either case the set $\eta^*_{i\nu}, \zeta^*_{i\nu}$ is a solution of (4.10) for the values of λ, k_i and h_θ indicated above, and the functions $\eta^*_{i\nu}$ satisfy (5.2) with all the functions $\eta_i{}^\beta$ $(\beta = 1, 2, \cdots, r_1 + \cdots + r_{s-1})$. Since the functions $\eta^0_{i\nu}(x)$ satisfy an inequality of the form (5.6) it is readily verified that the functions $\eta^*_{i\nu}$ satisfy a similar inequality. The functions $v_{i\nu} = \eta^*_{i\nu}$, together with the multipliers $\rho_{a\nu}$ and constants $d_{\gamma\nu}$ corresponding to the solution $\eta^*_{i\nu}, \zeta^*_{i\nu}$ of (4.10), give a solution of (5.4), (5.5) which satisfies the relations of Lemma 5.1.

Now let

$$(5.9)\qquad w_{i\nu}(x) = u_{i\nu}(x) + c v_{i\nu}(x) \qquad (i = 1, 2, \cdots, n;\ \nu = 1, 2, \cdots),$$

where c is a real constant. Then w_ν satisfies relations (5.2), and

$$I[w_\nu \mid \lambda_s] = I[u_\nu \mid \lambda_s] + c^2 I[v_\nu \mid \lambda_s] + 2c\big(H[u_\nu;\ v_\nu] - \lambda_s G[u_\nu;\ v_\nu]$$

$$+ \int_{x_1}^{x_2} \{u'_{i\nu} \Omega_{\eta'_i}(x, v_\nu, v'_\nu, \sigma_\nu) + u_{i\nu} \Omega_{\eta_i}(x, v_\nu, v'_\nu, \sigma_\nu) - \lambda_s u_{i\nu} K_{ij} v_{j\nu}\} dx\big).$$

When the integrals which occur in the coefficients of c and c^2 are integrated by parts in the usual manner and use is made of the fact that $v_{i\nu}, \sigma_{a\nu}$ is a solution of (5.4), (5.5) and that the functions u_ν are of class S_s, one obtains

$$(5.10)\qquad \begin{aligned} I[w_\nu \mid \lambda_s] &= I[u_\nu \mid \lambda_s] - 2c \\ &\quad - c^2 \{G[u_\nu;\ v_\nu] + \int_{x_1}^{x_2} u_{i\nu} K_{ij} v_{j\nu} dx\} \qquad (\nu = 1, 2, \cdots). \end{aligned}$$

From the above relation we obtain the following lemma, which is of use in proving the general Theorem 5.1.

LEMMA 5.2. *There exists a positive constant l such that for every admissible arc η we have*

$$I[\eta] \geqq l\{ \int_{x_1}^{x_2} \eta_i \eta_i dx + \eta_i(x_1)\eta_i(x_1) + \eta_i(x_2)\eta_i(x_2)\}.$$

The proof of this lemma depends upon the above relations for the special case in which $\| K_{ij} \|$ is the unit matrix and $2G[\eta(x_1), \eta(x_2)]$

$= \eta_i(x_1)\eta_i(x_1) + \eta_i(x_2)\eta_i(x_2)$, while $s = 1$. Let S_1^0 denote the above defined class S_1 corresponding to these particular values of $\| K_{ij} \|$ and $G[\eta]$. Lemma 5.2 then states that $\lambda_1^0 > 0$, where λ_1^0 is the greatest lower bound of $I[\eta]$ in the class S_1^0. Suppose that the lemma were not true. Since $\lambda_1^0 \geqq 0$ we would have $\lambda_\nu^0 = 0$, and there would exist a sequence of arcs u_ν^0 $(\nu = 1, 2, \cdots)$ of class S_1^0 such that $\lim_{\nu\to\infty} I[u_\nu^0] = 0$. Since (2.4), (2.5) is not compatible for $\lambda = 0$, we have by Lemma 5.1 that the solution $v^0_{i\nu}, \sigma^0_{a\nu}$ of (5.4), (5.5) for $\lambda = 0$ and corresponding to u_ν^0 for these particular values of $\| K_{ij} \|$ and $G[\eta]$, satisfies a relation

$$| v^0_{i\nu}(x) | \leqq l_3 \sum_{i=1}^n \{ \int_{x_1}^{x_2} (u^0_{i\nu})^2 \, dx + [u^0_{i\nu}(x_1)]^2 + [u^0_{i\nu}(x_2)]^2 \} + l_4,$$
$$\leqq l_3 + l_4,$$

where l_3 and l_4 are constants independent of x and ν. For $w^0_{i\nu}(x) = u^0_{i\nu}(x) + cv^0_{i\nu}(x)$ $(\nu = 1, 2, \cdots)$ we have from (5.10) that

$$I[w_\nu^0] = I[u_\nu^0] - 2c$$
$$- c^2 \sum_{i=1}^n \{u^0_{i\nu}(x_1)v^0_{i\nu}(x_1) + u^0_{i\nu}(x_2)v^0_{i\nu}(x_2) + \int_{x_1}^{x_2} u^0_{i\nu}v^0_{i\nu}dx\}$$

Now the coefficient of c^2 in this expression is in absolute value not greater than

$$\tfrac{1}{2} \sum_{i=1}^n \{[u^0_{i\nu}(x_1)]^2 + [u^0_{i\nu}(x_2)]^2$$
$$+ [v^0_{i\nu}(x_1)]^2 + [v^0_{i\nu}(x_2)]^2 + \int_{x_1}^{x_2} [(u^0_{i\nu})^2 + (v^0_{i\nu})^2] \, dx\}$$

and since u_ν^0 is of class S_1^0 and v_ν^0 satisfies the above inequality we have that there is a positive constant l_5 such that the coefficient of c^2 in the expression for $I[w_\nu^0]$ is in absolute value less than l_5. Since $\lim_{\nu\to\infty} I[u_\nu^0] = 0$, if we choose c such that $c^2 l_5 - 2c < 0$ it then follows that for ν sufficiently large we have $I[w_\nu^0] < 0$, which is impossible in view of (H4). Hence $\lambda_1^0 > 0$ and for $0 < l \leqq \lambda_1^0$ we have the inequality of Lemma 5.2 satisfied.

Let us now apply the results of these lemmas to prove the general Theorem 5.1. On the assumption that the theorem is false we have corresponding to each arc u_ν of (5.3) a solution $v_{i\nu}, \sigma_{a\nu}$ of (5.4), (5.5) satisfying the relations of Lemma 5.1, and finally for w_ν defined by (5.9) we have relation (5.10). Now clearly there exists a positive constant l_6 such that

$$| G[u_\nu; v_\nu] + \int_{x_1}^{x_2} u_{i\nu}K_{ij}v_{j\nu}dx | \leqq l_6 \sum_{i=1}^n \{u^2_{i\nu}(x_1) + u^2_{i\nu}(x_2)$$
$$+ v^2_{i\nu}(x_1) + v^2_{i\nu}(x_2) + \int_{x_1}^{x_2} [u^2_{i\nu}(x) + v^2_{i\nu}(x)] \, dx\}.$$

Since $\lim_{\nu\to\infty} I[u_\nu \mid \lambda_s] = \lim_{\nu\to\infty} \{I[u_\nu] - \lambda_s\} = 0$, it follows in view of 5.6 and Lemma 5.2 that there exists a positive constant l_7 such that the coefficient of c^2 in (5.10) is in absolute value less than l_7. Then, as in the proof of Lemma 5.2, we have that if c is chosen such that $c^2 l_7 - 2c < 0$ then for sufficiently large values of ν it follows that $I[w_\nu \mid \lambda_s] < 0$, which is impossible since λ_s was chosen as the greatest lower bound of $I[\eta]$ in the class S_s. Hence $\lambda = \lambda_s$ is a characteristic number of (2.4), (2.5) and $\lambda_s > \lambda_{s-1}$.

We shall now define the class S_{-1} as consisting of the totality of admissible arcs η such that

$$2G[\eta] + \int_{x_1}^{x_2} \eta_i K_{ij} \eta_j dx = -1, \tag{5.11}$$

and a sequence of classes $S_{-1}, S_{-2}, \cdots$ may be defined by induction in a manner analogous to that used in defining the sequence $S_1, S_2, \cdots$. If the matrix $\| K_{ij} \|$ is replaced by $\| -K_{ij} \|$ and $G[\eta]$ by $-G[\eta]$, then the classes $S_{-1}, S_{-2}, \cdots$ of the original problem correspond to the classes $S_1, S_2, \cdots$ for the modified problem. The following result then follows as a corollary to Theorem 5.1.

THEOREM 5.2. *If the class S_{-s} is not empty and $-\lambda_{-s}$ is the greatest lower bound of $I[\eta]$ in this class, then $\lambda = \lambda_{-s}$ is a negative characteristic number of* (2.4), (2.5) *and $\lambda_{-s} < \lambda_{-(s-1)}$.*

We have also the following property of the characteristic solutions of (2.4), (2.5).

LEMMA 5.3. *If g is an admissible arc and $G[\eta; g] + \int_{x_1}^{x_2} \eta_i K_{ij} g_j dx = 0$ for every solution η_i, μ_a of* (2.4), (2.5), *then $2G[g] + \int_{x_1}^{x_2} g_i K_{ij} g_j dx = 0$.*

For if the lemma were not true there would exist an admissible arc g satisfying the condition of the lemma with every solution η_i, μ_a of (2.4), (2.5) and $2G[g] + \int_{x_1}^{x_2} g_i K_{ij} g_j dx = l_8 \neq 0$; for definiteness, suppose $l_8 > 0$. There would then exist an arc η of class S_1 which satisfies the condition of the lemma, and for which $I[\eta] = l_9 > 0$. It readily follows that there would be infinitely many positive characteristic numbers of (2.4), (2.5) not greater than l_9, which is impossible in view of Lemma 4.1. A similar contradiction is obtained if we suppose that the arc g is such that $l_8 < 0$. Hence the lemma is proven.

From the results of this section it follows that each characteristic number

of the boundary value problem (2.4), (2.5) is uniquely determined as the minimum value of $I[\eta]$ in a corresponding class of admissible arcs.

6. Sufficient conditions for the existence of infinitely many characteristic numbers for (2.4), (2.5). On the assumption that the classes of functions $S_1, S_2, \cdots, S_s$ defined in § 5 are not empty it has been proven that corresponding characteristic numbers $0 < \lambda_1 < \cdots < \lambda_s$ exist for the boundary value problem (2.4), (2.5). If there exists an infinite sequence of classes $S_1, S_2, \cdots$ which are not empty there will clearly exist infinitely many positive characteristic numbers for (2.4), (2.5). A corresponding statement is true concerning the existence of infinitely many negative characteristic numbers. In this section will be given sufficient conditions for the boundary value problem (2.4), (2.5) to have infinitely many characteristic numbers.

Suppose that the matrix $\| K_{ij} \|$ and the differential equations (1.3) satisfy the additional condition:

($H5^+$) There is a sub-interval $x'_1x'_2$ of x_1x_2 such that if $\bar{x}_1$ and $\bar{x}_2$ are distinct points so that $x'_1 \leqq \bar{x}_1 < \bar{x}_2 \leqq x'_2$, then there exists a differentially admissible arc η which vanishes at $\bar{x}_1$ and $\bar{x}_2$ and such that $\int_{\bar{x}_1}^{\bar{x}_2} \eta_i K_{ij}\eta_j dx > 0$.

THEOREM 6.1. *If hypotheses* (H1), (H2), (H3), (H4) *and* ($H5^+$) *are satisfied, then the boundary value problem* (2.4), (2.5) *has infinitely many positive characteristic numbers.*

It will first be shown that the class S_1 is not empty. If $x'_1x'_2$ is a sub-interval which satisfies the condition of ($H5^+$), then there exists a differentially admissible arc $u \equiv (u_i)$ which vanishes at x'_1 and x'_2 and is such that $\int_{x'_1}^{x'_2} u_i K_{ij} u_j dx > 0$. If u is defined as equal to zero outside the sub-interval $x'_1x'_2$, then $2G[u] + \int_{x_1}^{x_2} u_i K_{ij} u_j dx > 0$ and therefore S_1 is not empty.

It will now be proven that if the classes $S_1, S_2, \cdots, S_{s-1}$ are not empty then the class S_s is also not empty. We shall use the notation of § 5. Let $R = r_1 + \cdots + r_{s-1}$ and divide $x'_1x'_2$ into $R+1$ consecutive intervals. By ($H5^+$) there exists an admissible arc $u_q \equiv (u_{iq})$ which vanishes outside the q-th interval and such that $\int_{x_1}^{x_2} u_{iq} K_{ij} u_{jq} dx > 0$. Then clearly constants $\delta_1, \delta_2, \cdots, \delta_{R+1}$ which are not all zero may be chosen such that the admissible arc defined as $u_i = \sum_{q=1}^{R+1} u_{iq}\delta_q$ satisfies the R linear relations $\int_{x_1}^{x_2} \eta_i^{\beta} K_{ij} u_j dx = 0$

$(\beta = 1, 2, \cdots, R)$. Now $2G[u] + \int_{x_1}^{x_2} u_i K_{ij} u_j dx > 0$, and therefore the class S_s is not empty. Theorem 6.1 is therefore established by induction.

Let $(\mathrm{H}5^-)$ denote the hypothesis obtained by replacing in $(\mathrm{H}5^+)$ the relation "$\int_{\bar{x}_1}^{\bar{x}_2} \eta_i K_{ij} \eta_j dx > 0$" by "$\int_{\bar{x}_1}^{\bar{x}_2} \eta_i K_{ij} \eta_j dx < 0$". This is equivalent to replacing $\| K_{ij} \|$ by $\| -K_{ij} \|$. We then have

COROLLARY. *If the hypotheses* (H1), (H2), (H3), (H4) *and* $(\mathrm{H}5^-)$ *are satisfied, then the boundary value problem* (2.4), (2.5) *has infinitely many negative characteristic numbers.*

Let (H2′) and (H3′) denote the following strengthened hypotheses:

(H2′) In addition to the matrix (2.1) being non-singular, the quadratic form

$$\omega_{\eta'_i \eta'_j}(x) \pi_i \pi_j \qquad x_1 \leqq x \leqq x_2$$

is non-negative for every set $(\pi_i) \neq (0)$ which satisfies the equations

$$\Phi_{\alpha \eta'_j} \pi_j = 0 \qquad (\alpha = 1, 2, \cdots, m).$$

(H3′) If $x'_1 x'_2$ is any sub-interval of $x_1 x_2$, then there do not exist functions $\mu_\alpha(x)$ not identically zero and constants c_i such that

$$\mu_\alpha(x) \Phi_{\alpha \eta'_i}(x) = \int_{x'_1}^{x} \mu_\alpha(t) \Phi_{\alpha \eta_i}(t) dt + c_i \quad x'_1 \leqq x \leqq x'_2 \quad (i = 1, 2, \cdots, n).$$

This assumption of normality on every sub-interval with respect to the differential equations (1.3) clearly implies (H3).

THEOREM 6.2. *If the system* (2.4), (2.5) *satisfies* (H1), (H2′), (H3′), *while the quadratic form $\eta_i K_{ij}(x) \eta_j$ is positive definite for each x on $x_1 x_2$ and $G[\eta]$ is non-negative, then for this boundary value problem there exist infinitely many positive characteristic numbers and only a finite number of negative characteristic numbers.*

If $x'_1 x'_2$ is any sub-interval of $x_1 x_2$, it follows from (H3′) that there are infinitely many differentially admissible arcs which vanish at x'_1 and x'_2 and are not identically zero on $x'_1 x'_2$.† Since $\eta_i K_{ij}(x) \eta_j$ is positive definite, condition $(\mathrm{H}5^+)$ is clearly satisfied. If (H4) is satisfied then Theorem 6.1 tells us that the boundary value problem has infinitely many positive characteristic numbers, and, since in this case the class S_{-1} is seen to be empty, there could be no negative characteristic numbers. If (H4) is not satisfied, however, in view of (H2′) and the condition that $\eta_i K_{ij}(x) \eta_j$ is positive

† See B. I, p. 689.

definite and $G[\eta]$ is non-negative, it follows that there exists a positive constant λ_0 such that the relation

$$0 \leqq I[\eta] + \lambda_0 \int_{x_1}^{x_2} \eta_i K_{ij} \eta_j dx \leqq I[\eta] + \lambda_0 \{2G[\eta] + \int_{x_1}^{x_2} \eta_i K_{ij} \eta_j dx\} \dagger$$

is satisfied by every admissible arc η, and the equality holds only if $\eta_i \equiv 0$ on x_1x_2. Theorem 6.2 then follows in view of the remark made at the end of § 2.

The denumerable set of characteristic solutions and characteristic numbers of the boundary value problem (2.4), (2.5) may be represented by the symbols $\eta_{is}(x)$, $\mu_{as}(x)$, λ_s $(s = 1, 2, \cdots)$. These solutions may also be chosen orthonormal in the sense that

$$G[\eta_s;\, \eta_t] + \int_{x_1}^{x_2} \eta_{is} K_{ij} \eta_{jt} dx = \delta_{st}(\operatorname{sgn} \lambda_s) \qquad (s, t = 1, 2, \cdots), \tag{6.1}$$

where $\delta_{st} = 0$ if $s \neq t$, $\delta_{ss} = 1$; $\operatorname{sgn} \lambda_s = 1$ if $\lambda_s > 0$, $\operatorname{sgn} \lambda_s = -1$ if $\lambda_s < 0$.

7. Expansion theorems. In this section there will be considered some expansion theorems in terms of the characteristic solutions of the boundary value problem (2.4), (2.5) *for the special case in which the coefficients of* $G[\eta(x_1), \eta(x_2)]$ *are zero. It will be assumed throughout this section that the hypotheses* (H1), (H2), (H3) *and* (H4) *are satisfied and that* (2.4), (2.5) *has infinitely many characteristic numbers.* The denumerably infinite set of characteristic solutions and characteristic numbers will be represented by $\eta_{is}(x)$, $\mu_{as}(x)$, λ_s $(s = 1, 2, \cdots)$ and we shall suppose that these solutions are orthonormal in the sense defined at the close of the last section; that is, such that

$$\int_{x_1}^{x_2} \eta_{is} K_{ij} \eta_{jt} dx = \delta_{st}(\operatorname{sgn} \lambda_s) \qquad (s, t = 1, 2, \cdots). \tag{7.1}$$

The canonical variables corresponding to the solution η_{is}, μ_{as}, and defined by (4.1), will be denoted by ζ_{is}.

We shall also make the following additional hypothesis:

(H6) If g is an arbitrary admissible arc and

$$\int_{x_1}^{x_2} \eta_{is} K_{ij} g_j dx = 0 \qquad (s = 1, 2, \cdots), \tag{7.2}$$

then $K_{ij}(x) g_j(x) \equiv 0$ on x_1x_2.

† See M, p. 533. The proof is given for $\| K_{ij} \|$ the identity matrix, but the same method applies to the case in which $\eta_i K_{ij} \eta_j$ is positive definite. The method used by Hickson (*loc. cit.*, p. 570) to establish the existence of such a value λ_0 for the special case of our general problem which he considered, may also be extended to obtain the above relation.

If the quadratic form $\eta_i K_{ij}\eta_j$ is positive definite on x_1x_2, then hypothesis (H6) is a consequence of the preceding hypotheses, and follows as a corollary to Lemma 5.3. We will give, however, another condition that will insure (H6).

LEMMA 7.1. *Hypothesis* (H6) *is satisfied by the characteristic solutions of* (2.4), (2.5) *whenever the following condition is satisfied by the differential system:*

(C) *If g is an arbitrary admissible arc and η_i, μ_a is the solution of the system*

$$\begin{aligned} &J_i(\eta,\mu) + K_{ij}g_j = 0, \qquad \Phi_a(x,\eta,\eta') = 0, \\ &L_{i1}(\eta,\mu,d,0) = 0 = L_{i2}(\eta,\mu,d,0) = \Psi_\gamma[\eta(x_1),\eta(x_2)], \end{aligned} \tag{7.3}$$

where L_{i1} and L_{i2} are defined by (2.5), *then $K_{ij}(x)\eta_j(x) \equiv 0$ on x_1x_2 only if $\eta_i \equiv 0 \equiv \mu_a$* on x_1x_2.

For if the lemma were false there would exist an admissible arc g satisfying the conditions (7.2), and such that the functions $K_{ij}(x)g_j(x)$ are not all identically zero on x_1x_2. For such an arc g the non-homogeneous system (7.3) would have a unique solution u_i, ρ_a since $\lambda = 0$ is not a characteristic number of (2.4), (2.5); furthermore, in view of condition (C), the functions $K_{ij}(x)u_j(x)$ are not all identically zero on x_1x_2. For such a solution u_i, ρ_a of (7.3) we would have in view of (3.3), (3.4′) and (3.5) that

$$\begin{aligned} \lambda_s \int_{x_1}^{x_2} u_i K_{ij}\eta_{js}\,dx &- \int_{x_1}^{x_2} \eta_{is}K_{ij}g_j\,dx \\ &= \int_{x_1}^{x_2} [\eta_{is}J_i(u,\rho) - u_iJ_i(\eta_s,\mu_s)]\,dx = 0 \quad (s = 1, 2, \cdots), \end{aligned} \tag{7.4}$$

and since $\lambda = 0$ is not a characteristic number of (2.4), (2.5) the arc u would also satisfy (7.2). If in the differential equations of (7.3) the arc g were replaced by the arc u, this system would have a unique solution v_i, σ_a and again, in view of (C), the functions $K_{ij}(x)v_j(x)$ are not all identically zero on x_1x_2; furthermore, the arc v would also satisfy the relations (7.2). Now set $w_i = v_i - u_i$ $(i = 1, 2, \cdots, n)$. The arc w would satisfy (7.2), and it would follow from Lemma 5.3 that

$$\begin{aligned} 0 &= \int_{x_1}^{x_2} [w_iK_{ij}w_j - u_iK_{ij}u_j - v_iK_{ij}v_j]\,dx \\ &= -2\int_{x_1}^{x_2} v_iK_{ij}u_j\,dx \end{aligned}$$

$$= 2 \int_{x_1}^{x_2} v_i J_i(v, \sigma)\, dx$$
$$= -2I[v],$$

and therefore, in view of (H4), $v_i \equiv 0$, which is a contradiction. Hence (C) is a sufficient condition to insure (H6).

Condition (C) is related to the condition that Bliss imposes in his definition of a definitely self-adjoint boundary value problem.† If (2.4), (2.5) is definitely self-adjoint then condition (C) is satisfied, but the converse is readily seen to not be true. The expansion theorems that we state will be related to the solutions of a non-homogeneous system of the form (7.3).

For an arbitrary arc g, define

$$(7.5) \qquad c_s[g] = \operatorname{sgn} \lambda_s \int_{x_1}^{x_2} g_i K_{ij} \eta_{js}\, dx \qquad (s = 1, 2, \cdots).$$

THEOREM 7.1. *If g is an arbitrary admissible arc such that the series*

$$(7.6) \qquad \sum_{s=1}^{\infty} c_s[g] \eta_{is}(x), \qquad \sum_{s=1}^{\infty} c_s[g] \zeta_{is}(x)$$

converge absolutely and uniformly on x_1x_2, and u_i, ρ_a is the solution of (7.3) corresponding to this arc g, then the series

$$(7.7) \qquad \sum_{s=1}^{\infty} c_s[u] \eta_{is}(x), \qquad \sum_{s=1}^{\infty} c_s[u] \zeta_{is}(x)$$

converge absolutely and uniformly on x_1x_2 to the values $u_i(x)$, $\Omega_{\eta'_i}[x, u(x), u'(x), \rho(x)]$.

From (7.4) it follows that

$$\lambda_s c_s[u] = c_s[g] \qquad (s = 1, 2, \cdots),$$

and therefore the series (7.7) converge absolutely and uniformly on x_1x_2. Now the set $g_i - \sum_{s=1}^{\infty} c_s[g] \eta_{is}(x)$ satisfies the relations (7.2), and therefore $K_{ij}\{g_j - \sum_{s=1}^{\infty} c_s[g] \eta_{is}\} \equiv 0$ on x_1x_2. It then follows that the set $u_i - \sum_{s=1}^{\infty} c_s[u] \eta_{is}$, $\Omega_{\eta'_i}(x, u, u', \rho) - \sum_{s=1}^{\infty} c_s[u] \zeta_{is}$ satisfies the system (4.10) for $\lambda = 0$, $k_i = 0$, $h_\theta = 0$, and is therefore identically zero. It also follows that the series $\sum_{s=1}^{\infty} c_s[u] \eta'_{is}(x)$ and $\sum_{s=1}^{\infty} c_s[u] \mu_{as}(x)$ converge absolutely and uniformly on x_1x_2 to the functions $u'_i(x)$, $\rho_a(x)$.

† See B. II, p. 570.

To prove a further expansion theorem for our system we make use of the following lemmas:

LEMMA 7.2. *If g is an admissible arc, then*

$$\sum_{s=1}^{\infty} c_s^2[g] \mid \lambda_s \mid \leqq I[g]. \tag{7.8}$$

This lemma follows from (H4) and the fact that if N is any positive integer one may prove directly that

$$I\{g - \sum_{s=1}^{N} c_s[g]\eta_s\} = I[g] - \sum_{s=1}^{N} c_s^2[g] \mid \lambda_s \mid.$$

LEMMA 7.3. *If g is an admissible arc and u_i, ρ_a is the solution of (7.3) corresponding to g, then the series (7.7) converge absolutely and uniformly on x_1x_2.*

It follows from (4.13) that

$$\eta_{is}(x) = -\lambda_s \int_{x_1}^{x_2} G^2{}_{il}(x,t)K_{lj}(t)\eta_{js}(t)\,dt, \tag{7.9}$$

$$\zeta_{is}(x) = -\lambda_s \int_{x_1}^{x_2} G^4{}_{il}(x,t)K_{lj}(t)\eta_{js}(t)\,dt, \tag{7.10}$$

where we have written $G^2{}_{il}(x,t)$ and $G^4{}_{il}(x,t)$ in place of $G^2{}_{il}(x,t,0)$ and $G^4{}_{il}(x,t,0)$. We then have in view of (4.12) that

$$\begin{aligned}\eta_{is}(x) &= \lambda_s^2 \int_{x_1}^{x_2} G^2{}_{il}(x,t)K_{lk}(t)\Big[\int_{x_1}^{x_2} G^2{}_{kr}(t,\xi)K_{rj}(\xi)\eta_{js}(\xi)\,d\xi\Big]\,dt \\ &= \lambda_s^2 \int_{x_1}^{x_2} \eta_{js}(\xi)K_{jr}(\xi)\Big[\int_{x_1}^{x_2} G^2{}_{rk}(\xi,t)K_{kl}(t)G^2{}_{li}(t,x)\,dt\Big]\,d\xi.\end{aligned}$$

In these expressions the subscripts l and r have also the range $1, 2, \cdots, n$. We likewise obtain

$$\zeta_{is}(x) = \lambda_s^2 \int_{x_1}^{x_2} \eta_{js}(\xi)K_{jr}(\xi)\Big[\int_{x_1}^{x_2} G^2{}_{rk}(\xi,t)K_{kl}(t)G^4{}_{il}(x,t)\,dt\Big]\,d\xi.$$

From Lemmas 4.2 and 7.2 we then obtain that for each fixed value of x the series

$$\sum_{s=1}^{\infty}\left(\frac{\eta_{is}(x)}{\lambda_s^2}\right)^2 \mid \lambda_s \mid, \qquad \sum_{s=1}^{\infty}\left(\frac{\zeta_{is}(x)}{\lambda_s^2}\right)^2 \mid \lambda_s \mid \tag{7.11}$$

converge; furthermore, from the explicit form (4.7) of the Green's matrix it is seen that there exists a constant l_{10} such that the series are less than l_{10}, uniformly for x on x_1x_2. Now since $\lambda_s c_s[u] = c_s[g]$, we have for each pair of integers N_1, N $(N_1 < N)$ the following relation

$$\sum_{s=N_1}^{N} c_s[u]\eta_{is}(x) = \sum_{s=N_1}^{N} \lambda_s \frac{\eta_{is}(x)}{\lambda_s^2} c_s[g]$$

and therefore

$$\left\{ \sum_{s=N_1}^{N} | c_s[u]\eta_{is}(x) | \right\}^2 \leqq \left\{ \sum_{s=N_1}^{N} | \lambda_s | \frac{| \eta_{is}(x) |}{\lambda_s^2} | c_s[g] | \right\}^2$$

$$\leqq \left\{ \sum_{s=N_1}^{N} | \lambda_s | \left(\frac{\eta_{is}(x)}{\lambda_s^2} \right)^2 \right\} \left\{ \sum_{s=N_1}^{N} | \lambda_s | c_s^2[g] \right\}$$

$$\leqq l_{10} \left\{ \sum_{s=N_1}^{N} | \lambda_s | c_s^2[g] \right\} .$$

A corresponding inequality is obtained for the second series of (7.11). It therefore follows in view of Lemma 7.2 that these series converge absolutely and uniformly and the lemma is proven.

Since the series (7.7) converge absolutely and uniformly on x_1x_2 the functions $u_i(x) - \sum_{s=1}^{\infty} c_s[u]\eta_{is}(x)$ satisfy the relations (7.2), and therefore $K_{ij}\{u_j - \sum_{s=1}^{\infty} c_s[u]\eta_{js}\} \equiv 0$ on x_1x_2. In particular, if $| K_{ij} | \neq 0$ on x_1x_2 we have $u_i(x) = \sum_{s=1}^{\infty} c_s[u]\eta_{is}(x)$ $(i = 1, 2, \cdots, n)$.

From Theorem 7.1 and Lemma 7.3 we now obtain the following expansion theorem.

THEOREM 7.2. *Let g be an arbitrary admissible arc and u_i, ρ_a the solution of (7.3) corresponding to g. If v_i, σ_a is the solution of the system*

$$J_i(v, \sigma) + K_{ij}u_j = 0, \qquad \Phi_a(x, v, v') = 0,$$
$$L_{i1}(v, \sigma, d, 0) = 0 = L_{i2}(v, \sigma, d, 0) = \Psi_\gamma[v(x_1), v(x_2)],$$

then the series

$$\sum_{s=1}^{\infty} c_s[v]\eta_{is}(x), \qquad \sum_{s=1}^{\infty} c_s[v]\zeta_{is}(x)$$

converge absolutely and uniformly on the interval x_1x_2 to the values $v_i(x)$, $\Omega_{\eta'_i}[x, v(x), v'(x), \sigma(x)]$. The series $\sum_{s=1}^{\infty} c_s[v]\eta'_{is}(x)$ and $\sum_{s=1}^{\infty} c_s[v]\mu_{as}(x)$ also converge absolutely and uniformly to the functions $v'_i(x)$, $\sigma_a(x)$.

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ON BOUNDARY VALUE PROBLEMS ASSOCIATED WITH DOUBLE INTEGRALS IN THE CALCULUS OF VARIATIONS.†

By William T. Reid.‡

1. *Introduction.* Lichtenstein § has considered a certain double integral problem with fixed boundary in the calculus of variations and has shown that the necessary condition of Jacobi is closely related to a certain boundary value problem associated with the Jacobi differential equation. If Jacobi's condition is satisfied, then the smallest positive characteristic number λ_1^+ of the boundary value problem which he considered must satisfy the inequality $\lambda_1^+ \geq 1$, and conversely. To establish a certain minimizing property of λ_1^+ he has made use of some expansion theorems for arbitrary functions in terms of the characteristic solutions of the associated boundary value problem. It is the purpose of the present paper to establish by the use of the methods of the calculus of variations this minimizing property of the first characteristic number in a much more elementary manner than that given by Lichtenstein; in particular, the existence of λ_1^+ is established by the methods of the calculus of variations. It is shown that in the associated boundary value problem the parameter may be allowed to enter in a simpler form than that used by Lichtenstein. By the use of this minimizing property of the first characteristic number, sufficient conditions for an extremal surface to render the double integral a weak relative minimum may be readily given.

2. *A boundary value problem.* Let A be a simply connected region in xy-space which is bounded by a closed analytic curve C. Let P, Q, R, B and K be functions of (x, y) which are analytic in their arguments in a region of xy-space which contains $A' + C$; it is also supposed that $PR - Q^2 > 0$, $P > 0$, and $B \leq 0$ on $A + C$. In this section will be considered the boundary value problem

$$(1) \qquad \partial(P\zeta_x + Q\zeta_y)/\partial x + \partial(Q\zeta_x + R\zeta_y)/\partial y + (B + \lambda K)\zeta = 0,$$

$$(2) \qquad \zeta = 0 \text{ on } C.$$

† Presented to the American Mathematical Society, December 31, 1930.

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§ L. Lichtenstein, *Monatshefte für Mathematik und Physik*, Vol. 28 (1917), pp. 3-51; also, *Mathematische Zeitschrift*, Vol. 6 (1919), pp. 26-51. These papers will be referred to as L. I and L. II respectively. See also, Picone, *Atti della Reale Accademia dei Lincei, Rendiconti* (5), Vol. 31_1 (1922), pp. 46-48.

A value λ will be said to be a characteristic number of this boundary value problem if corresponding to this value there exists a non-identically vanishing function $\zeta(x, y)$ which is of class C'' † on $A + C$, and satisfies the differential equation (1), together with the boundary condition (2).

For convenience, let

$$2\Omega(x, y, \zeta, \zeta_x, \zeta_y) = P\zeta_x^2 + 2Q\zeta_x\zeta_y + R\zeta_y^2 - B\zeta^2.$$

Then

$$2\Omega(x, y, \zeta, \zeta_x, \zeta_y) = \zeta_x\Omega_{\zeta_x} + \zeta_y\Omega_{\zeta_y} + \zeta\Omega_{\zeta},$$

$$\eta_x\Omega_{\zeta_x} + \eta_y\Omega_{\zeta_y} + \eta\Omega_{\zeta} = \zeta_x\Omega_{\eta_x} + \zeta_y\Omega_{\eta_y} + \zeta\Omega_{\eta}, \ddagger$$

and the differential equations (1) may be written

$$(1') \qquad \partial\Omega_{\zeta_x}/\partial x + \partial\Omega_{\zeta_y}/\partial y - \Omega_{\zeta} + \lambda K\zeta = 0.$$

If now $I_2(\eta, \zeta)$ is defined as the double integral

$$(3) \qquad I_2(\eta, \zeta) = \iint_A [\eta_x\Omega_{\zeta_x} + \eta_y\Omega_{\zeta_y} + \eta\Omega_{\zeta}]\, dx\, dy,$$

it follows after the usual integration by parts that

$$(4) \qquad I_2(\eta, \zeta) = \int_C \eta[\Omega_{\zeta_x} dy - \Omega_{\zeta_y} dx] - \iint_A \eta[\partial\Omega_{\zeta_x}/\partial x + \partial\Omega_{\zeta_y}/\partial y - \Omega_{\zeta}]\, dx\, dy.$$

If η and ζ are solutions of (1), (2) corresponding to distinct values λ and λ^*, since $I_2(\eta, \zeta) = I_2(\zeta, \eta)$, we have that

$$[\lambda - \lambda^*] \iint_A K\eta\zeta dx\, dy = 0.$$

Since $I_2(\zeta) \equiv I_2(\zeta, \zeta) > 0$ for every function ζ which satisfies (2) and is not identically zero on A, it is then readily seen that all the characteristic

† A function of (x, y) will be said to be of class $C^{(n)}$ $(n = 1, 2, \cdots)$ on $A + C$ if the function and all its partial derivatives of order not greater than n are continuous in the region A and on its boundary C. A function is said to be of class D' on $A + C$ if it is continuous and furthermore A may be divided into a finite number of sub-regions each of which is bounded by a simple closed curve consisting of a finite number of analytic pieces, and in each sub-region the given function is identical with a function which is of class C' in that sub-region and on its boundary.

‡ The derivatives of Ω are understood to have the arguments $(\zeta, \zeta_x, \zeta_y)$ or (η, η_x, η_y) as indicated by their subscripts.

numbers of the boundary value problem (1), (2) are real. If ζ is a solution of (1), (2) corresponding to a value λ, relation (4) gives

$$I_2(\zeta) = \lambda \iint_A K\zeta^2 \, dx \, dy, \tag{5}$$

and it is therefore seen that $\iint_A K\zeta^2 \, dx \, dy$ has the sign of λ. If then $K(x, y) \leq 0$ on $A + C$, the system (1), (2) can have no positive characteristic numbers. We will suppose in the following that $K(x, y) > 0$ for some point of A, and it will be shown that under this hypothesis there exists for (1), (2) a positive characteristic number.

Let H denote the class of functions $\zeta(x, y)$ which are of class D' on $A + C$ and vanish on C; H^* is then used to denote the class of functions ζ which belong to H and satisfy also the relation

$$\iint_A K\zeta^2 \, dx \, dy = 1. \tag{6}$$

On the hypothesis that K is positive at some point of A, the class H^* is not empty. Now since $I_2(\zeta) > 0$ for all functions ζ of class H^*, there exists a non-negative greatest lower bound λ_1 of the values of $I_2(\zeta)$ in the class H^*. Then the expression

$$I_2(\zeta \mid \lambda_1) = I_2(\zeta) - \lambda_1 \iint_A K\zeta^2 \, dx \, dy \tag{7}$$

is seen to be non-negative for all functions ζ of class H.

In this section will be established the following theorem:

THEOREM 2.1. *If λ_1 is the greatest lower bound of $I_2(\zeta)$ in the class H^*, then $\lambda = \lambda_1$ is a characteristic number of the boundary value problem* (1), (2).

In the proof of this theorem the following lemma will be used.

LEMMA 2.1. *There exists a positive constant α_1 such that if $\zeta(x, y)$ is any function of class D' on $A + C$ which vanishes on C, then*

$$\iint_A [\zeta_x^2 + \zeta_y^2] \, dx \, dy \geq \alpha_1 \iint_A \zeta^2 \, dx \, dy. \tag{8}$$

For let S be a square which contains A in its interior and whose corners have the coördinates (x_0, y_0), $(x_0 + d, y_0)$, $(x_0 + d, y_0 + d)$, $(x_0, y_0 + d)$,

where $d > 0$. Let $\zeta(x, y)$ be defined as zero outside of A. Then by Schwarz' inequality,

$$[\zeta(x,y)]^2 = [\int_{x_0}^{x} \zeta_x(x,y)dx]^2 \leq [x - x_0] \int_{x_0}^{x} \zeta_x^2(x,y)\,dx$$
$$\leq [x - x_0] \int_{x_0}^{x_0+d} \zeta_x^2(x,y)\,dx$$

and therefore

$$\iint_A \zeta^2 dxdy = \iint_S \zeta^2 dxdy \leq (d^2/2) \iint_S \zeta_x^2 dxdy = (d^2/2) \iint_A \zeta_x^2 dxdy.$$

Inequality (8) then follows by combining this inequality and a similar one for $\iint_A \zeta_y^2\, dx\, dy$.

Theorem 2.1 will now be proved. The equation (1) for $\lambda = \lambda_1$ is the Euler-Lagrange equation for the problem of minimizing $I_2(\zeta \mid \lambda_1)$ in the class H. Since $I_2(\zeta \mid \lambda_1) \geq 0$, we have that $\zeta(x, y) \equiv 0$ is a minimizing surface for this integral, and therefore for it the necessary condition of Jacobi must be satisfied. That is, if $\bar{A}$ is a subregion of A bounded by a simply closed curve $\bar{C}$ which consists of a finite number of analytic pieces and lies in A, then there exists no solution of the Jacobi differential equation which vanishes on $\bar{C}$ and is not identically zero on $\bar{A}$.† Since the integrand of $I_2(\zeta \mid \lambda_1)$ is quadratic in ζ, ζ_x and ζ_y, we have that the Jacobi equation is identical with the Euler-Lagrange equation and is given by (1) with $\lambda = \lambda_1$.

Now if $\lambda = \lambda_1$ is not a characteristic number of (1), (2) there exists a unique solution $u(x, y)$ of (1) for $\lambda = \lambda_1$ such that $u(x, y) = 1$ on C.‡

† See Bolza, *Variationsrechnung*, Leipzig, 1909, p. 675. The proof of this condition depends upon the existence of an elementary solution of the Jacobi equation. The existence of an elementary solution for the general equation of elliptic type whose coefficients are analytic has been proven by Hadamard. See J. Hadamard, *Lectures on Cauchy's Problem*, London, 1923. The existence of an elementary solution for the general equation of elliptic type whose coefficients satisfy much weaker conditions has been established by Levi. See E. E. Levi, *Rendiconti del Circolo Matematico di Palermo*, Vol. 24 (1907), pp. 275-317.

‡ See L. I, p. 16. Lichtenstein shows that the boundary value problem (1), (2) may be reduced to one associated with the normal form of equation (1). The system (1), (2) may also be considered directly and it may be shown that if for $\lambda = \lambda_1$ there is no solution of the boundary value problem, then there exists a unique solution of (1) which coincides on the boundary C with an arbitrarily chosen analytic function. For a treatment of the corresponding problem for the general linear elliptic differential equation of the second order with three independent variables, see W. Sternberg,

It will now be shown that $u(x,y) > 0$ on A. For if at a point of A this solution $u(x,y)$ vanishes, then there are points of A at which $u(x,y) < 0$.† From this it follows that there is a sub-region $\bar{A}$ of A in which $u(x,y)$ is not identically zero and on the boundary $\bar{C}$ of $\bar{A}$ we have $u(x,y) = 0$. Furthermore, $\bar{C}$ consists of a finite number of analytic pieces.‡ But since λ_1 is the greatest lower bound of $I_2(\zeta)$ in the class H^*, this is impossible, as shown above. It therefore follows that $u(x,y) > 0$ on $A + C$.

If the transformation of Clebsch is then applied to $I_2(\zeta \mid \lambda_1)$, it is seen that for an arbitrary function $\zeta(x,y)$ of class H,§

$$I_2(\zeta \mid \lambda_1) = \iint_A u^2 \left[P\bar{\zeta}_x{}^2 + 2Q\bar{\zeta}_x\bar{\zeta}_y + R\bar{\zeta}_y{}^2\right] dx\, dy, \tag{9}$$

where $\bar{\zeta} = \zeta/u$. Since $u(x,y) > 0$ on $A + C$, and $PR - Q^2 > 0$ on $A + C$, there is a positive constant α_2 such that

$$I_2(\zeta \mid \lambda_1) \geqq \alpha_2 \iint_A \left[\bar{\zeta}_x{}^2 + \bar{\zeta}_y{}^2\right] dx\, dy,$$

and therefore, in view of lemma 2. 1,

$$I_2(\zeta \mid \lambda_1) \geqq \alpha_1\alpha_2 \iint_A \bar{\zeta}^2\, dx\, dy.$$

Now

$$\bar{\zeta}^2 \geqq [\max(u)]^{-1}\zeta^2 \geqq [\max(u)]^{-1}[\max(K)]^{-1}K\zeta^2,$$

and therefore there is a positive constant α_3 such that

$$I_2(\zeta \mid \lambda_1) \geqq \alpha_3 \iint_A K\zeta^2\, dx\, dy.$$

But this inequality, which has been obtained on the assumption that λ_1 is not a characteristic number of (1), (2), is a contradiction to the hypothesis that λ_1 is the greatest lower bound of $I_2(\zeta)$ in the class H^*. Hence $\lambda = \lambda_1$

Mathematische Zeitschrift, Vol. 21 (1924), pp. 286-311. The coefficients of the equation treated by Sternberg are supposed to satisfy a condition much weaker than being analytic; in fact, if they are of class C''' then the condition imposed is satisfied.

† When the coefficients of (1) are analytic, as supposed above, this follows from results established by Picard. See Picard, *Traité d'Analyse*, Paris, 1905, Vol. 2, pp. 28-32. The result is also true when the coefficients of the differential equation are not required to be analytic; see L. Lichtenstein, *Rendiconti del Circolo Matematico di Palermo*, Vol. 33 (1912), pp. 201-211.

‡ See L. I, p. 10. The proof is there indicated for the normal form of (1), but the same proof may be applied directly to the differential equation (1).

§ Bolza, *loc. cit.*, p. 680.

is a characteristic number for the boundary value problem (1), (2). Furthermore, since (5) holds for every solution of (1) corresponding to a value λ, we have that $\lambda_1 > 0$ and that $\lambda = \lambda_1$ is the smallest positive characteristic number of (1), (2).

3. *The calculus of variations problem.* As in § 2, let A be a simply connected region in the xy-plane whose boundary C is a simple closed analytic curve. Consider the problem of minimizing the integral

$$I = \iint_A f(x, y, z, z_x, z_y)\, dx\, dy \tag{10}$$

in the class of surfaces

$$z = z(x, y) \tag{11}$$

which are of class D' on $A + C$ and which coincide with a given function $\psi(x, y)$ on the boundary C. The function $\psi(x, y)$ is supposed to be an analytic function of the arc element on C, and any surface z of the type described above will be said to be *admissible*.

Suppose that E is an admissible surface and in a neighborhood $\mathfrak{R}$ of the values (x, y, z, z_x, z_y) on E the function $f(x, y, z, p, q)$ is an analytic function of its five arguments. It is also supposed that E is an extremal surface, that is, it is of class C'' and satisfies the Euler-Lagrange differential equation

$$\partial f_p/\partial x + \partial f_q/\partial y - f_z = 0 \tag{12}$$

on $A + C$; furthermore, the Legendre condition is satisfied in the strong form, that is,

$$f_{pp}f_{qq} - f^2_{pq} > 0, \qquad f_{pp} > 0 \qquad [(x, y) \text{ on } A + C]. \tag{13}$$

In the partial derivatives of f which occur in (12) and (13) it is understood that the arguments $(x, y, z(x, y), z_x(x, y), z_y(x, y))$ are those which belong to E. According to Lichtenstein † the extremal surface $z = z(x, y)$ is then an analytic function of (x, y) on $A + C$.

The second variation of I along E is then given by

$$I_2(\zeta) = \iint_A 2\omega(x, y, \zeta, \zeta_x, \zeta_y)\, dx\, dy, \tag{14}$$

where $$2\omega(x, y, \zeta, \zeta_x, \zeta_y) = f_{pp}\zeta_x^2 + 2f_{pq}\zeta_x\zeta_y + f_{qq}\zeta_y^2 + 2f_{pz}\zeta_x\zeta + 2f_{qz}\zeta_y\zeta + f_{zz}\zeta^2.$$

† L. Lichtenstein, *Bulletin international de l'Académie des Sciences de Cracovie (Classe des sciences mathématiques et naturelles)* 1912, pp. 915-936.

A function $\zeta = \zeta(x, y)$ which is of class D' on $A + C$ and vanishes on C will be said to be an *admissible variation.* If $\zeta(x, y)$ is of class C'' on $A + C$, then by an integration by parts we obtain

$$I_2(\zeta) = -\iint_A \zeta J(\zeta)\, dx\, dy, \tag{15}$$

where

$$J(\zeta) = \partial\omega_{\zeta_x}/\partial x + \partial\omega_{\zeta_y}/\partial y - \omega. \tag{16}$$

It is seen that

$$J(\zeta) = L(\zeta) + k\zeta,$$

where

$$\begin{aligned} L(\zeta) &= \partial(f_{pp}\zeta_x + f_{pq}\zeta_y)/\partial x + \partial(f_{pq}\zeta_x + f_{qq}\zeta_y)/\partial y, \\ k(x, y) &= \partial f_{pz}/\partial x + \partial f_{qz}/\partial y - f_{zz}. \end{aligned} \tag{17}$$

If the terms of 2ω which contain $\zeta\zeta_x$ and $\zeta\zeta_y$ are integrated by parts, one obtains

$$I_2(\zeta) = \iint_A [f_{pp}\zeta_x^2 + 2f_{pq}\zeta_x\zeta_y + f_{qq}\zeta_y^2 - k\zeta^2]\, dx\, dy. \tag{14'}$$

Since $I_2(\zeta) \geqq 0$ for every admissible variation if E is a minimizing surface, we have: *If E is a minimizing surface for I, then there are no negative characteristic numbers for the boundary value problem*

$$J(\zeta) + \lambda\zeta = 0, \qquad \zeta = 0 \text{ on } C. \tag{18}$$

This follows immediately since if ζ is a solution of (18) corresponding to a value λ, then

$$I_2(\zeta) = \lambda \iint_A \zeta^2 dx\, dy.$$

The coefficients of the boundary value problem (18) as given above do not necessarily satisfy the conditions imposed on the coefficients of (1) in § 2, but if a positive number $\bar{\lambda}$ is chosen such that $|\, k(x, y)\, | \leqq \bar{\lambda}$ on $A + C$ and λ is replaced by $\mu = \lambda + \bar{\lambda}$, then (18) becomes

$$L(\zeta) + [(k - \bar{\lambda}) + \mu]\zeta = 0, \qquad \zeta = 0 \text{ on } C, \tag{18'}$$

and the coefficients of (18′) satisfy the conditions imposed in § 2. The boundary value problem (18′) is seen to have only positive characteristic numbers μ, and by § 2 there exists a smallest characteristic number μ_1. Then $\lambda_1 = \mu_1 - \bar{\lambda}$ is the smallest characteristic number of (18) and we have seen that if E is a minimizing surface, then $\lambda_1 \geqq 0$.

The following sufficiency condition will now be proved.

THEOREM 3.1. *If $z = z(x, y)$ is an analytic extremal surface along which the condition of Legendre is satisfied in the strong form and for which the smallest characteristic number of the boundary value problem* (18) *is positive, then z renders the integral I a weak relative minimum.*

For let $\zeta(x, y)$ be an arbitrary admissible variation such that the surface $z(x, y) + \zeta(x, y)$ lies in the neighborhood $\mathfrak{N}$ of the surface z. Then

$$\Delta I = \iint_A f(x, y, z + \zeta, z_x + \zeta_x, z_y + \zeta_y)\, dx\, dy$$

$$- \iint_A f(x, y, z, z_x, z_y)\, dx\, dy$$

$$= \tfrac{1}{2} \iint_A 2\bar{\omega}(x, y, \zeta, \zeta_x, \zeta_y)\, dx\, dy,$$

where $2\bar{\omega}$ is a quadratic form in ζ, ζ_x, ζ_y whose coefficients are

$$\bar{f}_{pp} = 2 \int_0^1 (1 - \theta) f_{pp}(x, y, z + \theta\zeta, z_x + \theta\zeta_x, z_y + \theta\zeta_y)\, d\theta, \text{ etc.}$$

We will now say that an admissible variation ζ belongs to the class $R[\delta]$ if the functions $|\zeta|$, $|\zeta_x|$ and $|\zeta_y|$ are less than δ, uniformly on $A + C$. From the form of $2\bar{\omega}$ it is seen that for every positive ϵ there exists a positive δ_ϵ such that if ζ belongs to $R[\delta_\epsilon]$, then the coefficients of the quadratic form $2\bar{\omega} - 2\omega$ are all in absolute value less than ϵ, uniformly on $A + C$. Now apply the inequality $2ab \leq a^2 + b^2$ to each of the cross-product terms in the quadratic form $2\bar{\omega} - 2\omega$. By the use of Lemma 2.1, together with the fact that from (13) it follows that there is a positive constant α_4 such that

$$\zeta_x^2 + \zeta_y^2 \leq \alpha_4[f_{pp}\zeta_x^2 + 2f_{pq}\zeta_x\zeta_y + f_{qq}\zeta_y^2],$$

it is seen that if ζ belongs to $R[\delta_\epsilon]$, then

$$(22) \qquad \iint_A |\, 2\omega(x, y, \zeta, \zeta_x, \zeta_y) - 2\bar{\omega}(x, y, \zeta, \zeta_x, \zeta_y)\,|\, dx\, dy$$

$$\leq \epsilon\alpha_5 \iint_A [f_{pp}\zeta_x^2 + 2f_{pq}\zeta_x\zeta_y + f_{qq}\zeta_y^2]\, dx\, dy,$$

where $\alpha_5 = 3\alpha_4(1 + 1/\alpha_1)$. Let $\bar{\lambda}$ be a positive constant such that $|k(x, y)| \leq \bar{\lambda}$ on $A + C$. It then follows from (14′) that if ζ belongs to $R[\delta_\epsilon]$, then

$$\left| \iint_A [2\omega(x, y, \zeta, \zeta_x, \zeta_y) - 2\bar{\omega}(x, y, \zeta, \zeta_x, \zeta_y)] dx\, dy \right|$$
$$\leq \epsilon\alpha_5 \iint_A [2\omega(x, y, \zeta, \zeta_x, \zeta_y) + \bar{\lambda}\zeta^2] dx\, dy$$

and therefore

$$(23) \qquad \Delta I \geq \tfrac{1}{2}\{(1 - \epsilon\alpha_5) \iint_A 2\omega(x, y, \zeta, \zeta_x, \zeta_y) dx\, dy - \epsilon\alpha_5\bar{\lambda} \iint_A \zeta^2 dx\, dy\}.$$

If now the smallest characteristic number λ_1 of (18) is positive, one may choose ϵ' such that

$$(1 - \epsilon'\alpha_5)\lambda_1 - \epsilon'\alpha_5\lambda > 0,$$

and, furthermore, such that if ζ belongs to $R[\delta_{\epsilon'}]$, then the surface $z(x, y) + \zeta(x, y)$ lies in the neighborhood $\mathfrak{R}$ of the surface z. In view of the minimizing property of λ_1, it follows from (23) that if $\zeta(x, y)$ is an admissible variation which belongs to $R[\delta_{\epsilon'}]$, then

$$(24) \qquad \Delta I \geq \tfrac{1}{2}[(1 - \epsilon'\alpha_5)\lambda_1 - \epsilon'\alpha_5\bar{\lambda}] \iint_A \zeta^2 dx\, dy,$$

and therefore for such variations ζ we have $\Delta I \geq 0$ and $\Delta I = 0$ only if $\zeta(x, y) \equiv 0$ on A. Theorem 3. 1 is therefore proved.

Instead of the boundary value problem (18) one may consider the boundary value problem

$$(25) \qquad L(\zeta) + g\zeta + \lambda(k - g)\zeta = 0, \qquad \zeta = 0 \text{ on } C,$$

where $k(x, y)$ is defined by (17) and $g(x, y)$ is an arbitrary function which is analytic and non-positive on $A + C$; in terms of this boundary value problem one obtains then a sufficient condition for a weak relative minimum, analogous to Theorem 3. 1.

The boundary value problem (25) is the one considered by Lichtenstein. He treated the particular case where $g(x, y) \equiv 0$, but he stated that results analogous to those established for this special case are true for the general problem (25). Since E is supposed to be an analytic extremal surface and the function $f(x, y, z, p, q)$ is analytic in its arguments in a neighborhood $\mathfrak{R}$ of the values corresponding to E, and since $g(x, y)$ is supposed to be analytic and non-positive on $A + C$, the coefficients of the boundary value problem (25) satisfy the conditions imposed in § 2.

From the expression analogous to (5) we have that if ζ is a solution of (25) corresponding to a value λ, then λ and $\iint_A (k - g)\zeta^2 dx\, dy$ have the

same sign; furthermore, for such a solution ζ,

$$\iint_A 2\omega(x, y, \zeta, \zeta_x, \zeta_y)\,dx\,dy = (\lambda - 1)\iint_A (k - g)\zeta^2 dx\,dy.$$

From this relation, we have: *If E is a minimizing surface for I, then there exists no positive characteristic number of* (25) *which is less than unity.*

Corresponding to Theorem 3.1 we have the following sufficiency theorem.

THEOREM 3.2. *If $z = z(x, y)$ is an analytic extremal surface along which the condition of Legendre is satisfied in the strong form and for which the boundary value problem* (25) *has no positive characteristic number which is not greater than unity, then the surface z renders the integral I a weak relative minimum.*

For if $(k - g) > 0$ at a point of A, it then follows from § 2 that there is a first positive characteristic number λ_1^+ of (25), and for all admissible variations ζ we have the relation

$$\iint_A [f_{pp}\zeta_x^2 + 2f_{pq}\zeta_x\zeta_y + f_{qq}\zeta_y^2 - g\zeta^2]\,dx\,dy - \lambda_1^+ \iint_A (k - g)\zeta^2\,dx\,dy \geq 0. \tag{26}$$

From (14′) and (26) it then follows that

$$\iint_A 2\omega(x, y, \zeta, \zeta_x, \zeta_y)\,dx\,dy \geq (1 - 1/\lambda_1^+)\iint_A [f_{pp}\zeta_x^2 + 2f_{pq}\zeta_x\zeta_y + f_{qq}\zeta_y^2 - g\zeta^2]\,dx\,dy. \tag{27}$$

Since $g(x, y) \leq 0$, it follows from (22) that if ζ belongs to $R[\delta_\epsilon]$, then

$$\Delta I \geq \tfrac{1}{2}(1 - 1/\lambda_1^+ - \epsilon\alpha_5)\iint_A [f_{pp}\zeta_x^2 + 2f_{pq}\zeta_x\zeta_y + f_{qq}\zeta_y^2 - g\zeta^2]\,dx\,dy. \tag{28}$$

If $\lambda_1^+ > 1$, then ϵ' may be chosen so small that $1 - 1/\lambda_1^+ - \epsilon'\alpha_5 > 0$ and also such that if $\zeta(x, y)$ belongs to $R[\delta_{\epsilon'}]$ then the surface $z(x, y) + \zeta(x, y)$ lies in the neighborhood $\mathfrak{N}$ of the surface $z(x, y)$. It then follows that if $\zeta(x, y)$ is an admissible variation which belongs to $R[\delta_{\epsilon'}]$ then $\Delta I \geq 0$ and the equality sign holds only if $\zeta(x, y) \equiv 0$ on A. Theorem 3.2 is therefore proved in case $(k - g) > 0$ at some point of A. If $(k - g) \leq 0$ on A, then

relations (27) and (28) hold if $1/\lambda_1^+$ is replaced by zero, and the surface z is seen to render the integral I a weak relative minimum.

Lichtenstein obtained the result stated in Theorem 3. 2. The proof given by Lichtenstein is much more complicated than the above, however, since use is made of expansion theorems in terms of the characteristic solutions corresponding to the infinite set of characteristic numbers for the boundary value problem.† Instead of using so much from the theory of elliptic partial differential equations, it has been shown in § 2 by the use of methods of the calculus of variations that there *exists* a *first* characteristic number for this boundary value problem, and this first characteristic number is characterized by its minimizing property.

It is to be mentioned that Theorem 3. 1 may be deduced as a corollary of Theorem 3. 2. For let $\bar{\lambda}$ be a positive constant such that $|k(x,y)| \leq \bar{\lambda}$ on $A + C$, and define $g(x,y)$ as equal to $k(x,y) - \bar{\lambda}$. For this choice of $g(x,y)$ the boundary value problem (25) reduces to

$$(25^*) \qquad J(\zeta) + (\lambda - 1)\bar{\lambda}\zeta = 0, \qquad \zeta = 0 \text{ on } C.$$

If λ^*_1 is the smallest positive characteristic number of (25*), then $\lambda^*_1 \geq 1$ $[\lambda^*_1 > 1]$ if and only if $\lambda_1 \geq 0$ $[\lambda_1 > 0]$, where λ_1 is the smallest characteristic number of (18). However, since the parameter λ enters in such a simple manner in (18), it seems worthwhile to consider this boundary value problem independent of the problem (25).

Lichtenstein ‡ has also shown that if the double integral problem is regular and $z = z(x,y)$ is an admissible extremal surface for which the smallest positive characteristic number of the boundary value problem (25) is greater than unity, then the extremal surface $z(x,y)$ may be imbedded in a field of extremal surfaces. In the proof of this result use is made of the minimizing property of λ_1^+ which has been established in § 2. One may then deduce sufficient conditions for a strong relative minimum by the method of Weierstrass.§

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† See L. I and L. II; in particular, L. II, pp. 29-34.
‡ See L. I and L. II; in particular, L. II, pp. 34-40.
§ See Bolza, *loc. cit.*, p. 683.

NOTE ON A GENERALIZATION OF THE LAGRANGE-GAUSS MODULAR ALGORITHM.

By Aurel Wintner.

According to a communication of Professor H. Geppert, the algorithm treated in a previous note * may be reduced to the one which belongs to $\rho = 1$, finding

$$M_\rho(a, b)^\rho = M_1(a^\rho, b^\rho).$$

From the classical properties of M_1 there follow, therefore, the answers to the questions raised but not solved in that note, and, in particular, the numerical values of the limits which correspond to the classical limit of Gauss (cf. p. 352, *loc. cit.*).

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* A. Wintner, "On a Generalization of the Lagrange-Gauss Modular Algorithm," *American Journal of Mathematics*, Vol. 54 (1932), pp. 346-352.